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A general nonlinear programming model of a producers cooperative association in the short-run

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CHAPTER I. INTRODUCTION

Problem Statement

A producers cooperative association is an organization of firms each of which contributes some of its resources to a jointly-owned enterprise, or cooperative, which processes and markets output of member patrons or supplies them with inputs which they use in production. In addition, a cooperative may process and market output of nonmember patrons or supply them with inputs which they use in production.

In analyzing the cooperative association, it is necessary to distinguish between the cooperative association and the economic entities of which it is composed. The cooperative association consists of its member firms, including both the resources which they individually own and control and those resources which they jointly own and which are controlled by the cooperative. The cooperative consists of these jointly-owned resources. With respect to the member firms, there is a distinction between ownership and control. Each firm consists of the resources which it individually owns and controls and a share of those which it jointly owns with the other member firms and which are controlled by the cooperative.

The organizational structure of the cooperative association and its member firms is illustrated in Figure 1.1. Each of the wedges represents a member firm while the area within the circle represents the cooperative. The resources of a member firm which are jointly held and controlled are represented by that portion of the corresponding wedge within

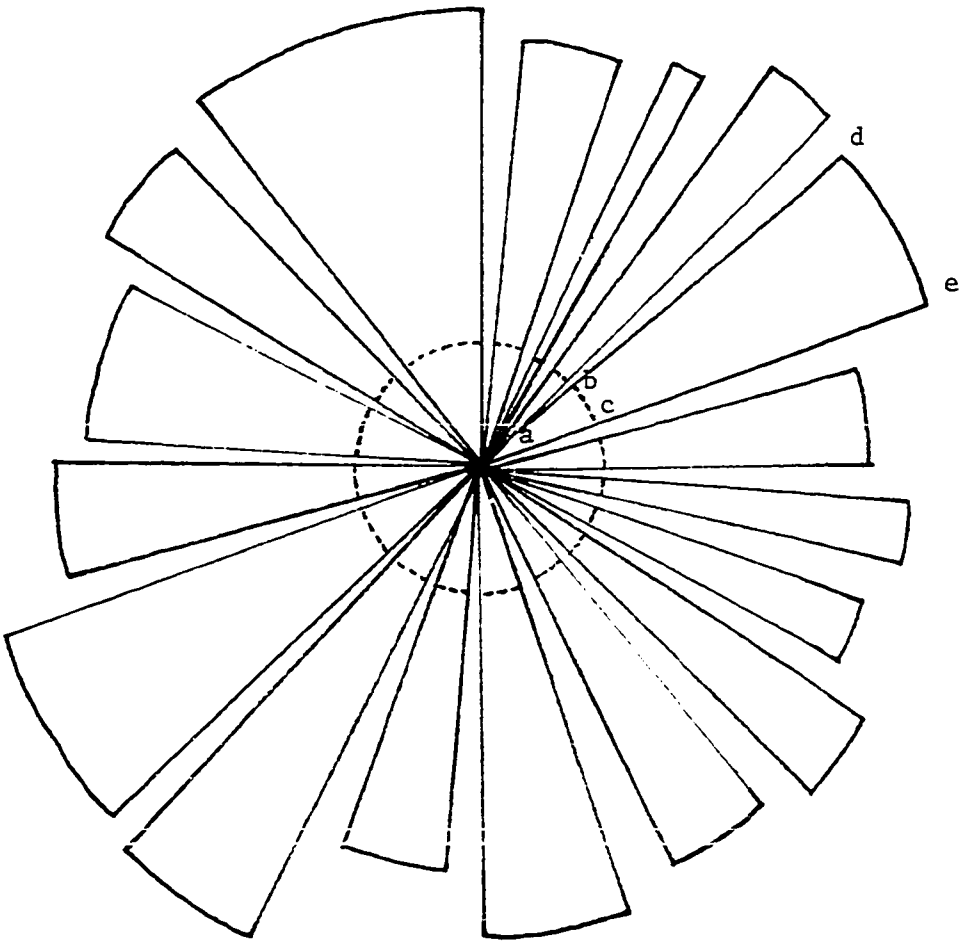

Figure 1.1. The cooperative association (adapted from Phillips [44, Fig. 1, p. 76]).

the circle, such as abc. That portion outside the circle, such as area bdec, represents those resources of the member firm which are outside the cooperative and which are individually controlled. The size of the member firm is represented by the size of the corresponding wedge. The extent of its patronage with the cooperative is represented by the wedge's width. As the figure indicates, there is no relationship between member firm size and patronage. However, as the figure does indicate, the contribution of resources to the cooperative is, at least ideally, in proportion to patronage. The extent to which this relationship exists in reality is due largely to the fact that much of the capital of the cooperative is gathered from patronage refunds and per-unit capital investments.

Within the cooperative association, there are not one, but many decision-making units. The entrepreneur or decision-maker for each member firm must make decisions concerning which products the firm will produce, how much of each of these products it will produce, what production methods it will use, and the quantity of each of the factors of production it will need. Just as the decision-maker of each of the member firms must make decisions concerning its operation, a decision-maker for the cooperative must make decisions concerning the operation of the cooperative.

This decision-maker may be the manager of the cooperative or the cooperative's board of directors. It may also be a group of member firms or the cooperative association as a whole. The purpose of this study is not to identify the individual or individuals who make the decisions for

the cooperative. Therefore, throughout this study, the identity of the cooperative decision-maker is left unspecified. Instead, it is only assumed that there is a cooperative decision-maker who makes decisions for the cooperative according to a cooperative objective function, just as the decision-makers for the member firms make decisions for these firms according to their objective functions.

Some of the decisions which must be made by the cooperative decision-maker are of a long-run nature. These include decisions on investment and financing and are beyond the scope of this study. This study is only concerned with the decisions which must be made by the cooperative decision-maker in the short-run, the period of time in which the resources of the cooperative are fixed. As it is, there are a number of decisions involving the operation of the cooperative in the short-run which must be made by the cooperative decision-maker. There are also a number of problems and issues concerning these decisions.

One of these problems is that of identifying the cooperative's objective or objectives. An objective is necessary if the cooperative decision-maker is to operate the cooperative in a rational manner. Several possible objectives have been suggested in the literature on cooperative associations, but there appears to be no agreement on what the objective or objectives of the cooperative are. There is some agreement that the purpose of the cooperative is to benefit its members. If the cooperative decision-maker attempts to benefit the members of the cooperative association, the objective or objectives of the cooperative cannot be independent of the objectives of the member firms and the

welfare of the members must be an argument in the cooperative objective function. The proprietary firm is generally not considered to be interested in how its actions affect the ability of those with whom it trades to achieve their objectives. On the other hand, a cooperative decision-maker who attempts to benefit the cooperative association's member firms must be concerned with how the cooperative's actions affect the ability of the member firms to achieve their objectives.

It is not always clear, however, what is meant by a cooperative "benefitting its members." If a decision made by a cooperative decision-maker results in an increase in the value of the objective function of each and every one of the cooperative association's member firms, the action can probably be said to benefit the member firms. However, a decision made by the cooperative decision-maker may make some members better off while making others worse off. Similarly, a decision made by the cooperative decision-maker may attract new members to the cooperative association, but make some existing members worse off. Thus, selection of a cooperative objective function involves making decisions concerning the distribution of benefits among members and between existing and potential members.

Assuming that the cooperative decision-maker is successful in defining the objective or objectives of the cooperative, he must be able to make decisions which will result in an efficient operation of the cooperative. In other words, he must make decisions which will result in the maximum value of the cooperative objective function. To do this, the cooperative decision-maker must determine which variables can be used as

instruments in achieving the cooperative's objective or objectives. The cooperative decision-maker's choice of instruments depends upon what the cooperative's objective or objectives are. The choice may also depend upon the cooperative's operating principles. For example, for a supply cooperative, the use of prices as instruments may be greatly limited if the cooperative decision-maker feels the cooperative must supply all of its goods and services to members at cost.

Among the decisions which the cooperative decision-maker must make, are a number of production and pricing decisions. The cooperative decision-maker must determine which goods and services the cooperative will supply its member firms. He must determine how much of each of these products it will produce, what production processes it will use, and the quantity of each of the factors of production it will require. If the cooperative consists of more than one plant, the cooperative decision-maker must make these decisions for each of the plants. In addition, he must determine the prices the cooperative will charge members for the goods and services it supplies them and the prices it will offer them for the goods they market through it.

Some of these production and pricing decisions may involve goods and services which are public goods or which are characterized by externalities. Within the cooperative association, a public good is a good or service provided by the cooperative which benefits all member firms in such a manner that the benefits received by one member firm do not diminish those received by any other. There are two types of public goods which may be provided by the cooperative. There are those which

affect production, and there are those which affect price. The former include research and development and the dissemination of production information. The latter include bargaining services and advertising. Within the cooperative association, an externality is a benefit or spill-over to one member firm resulting from the use or production of a good or service by another. Disease control is an example of a good which is characterized by externalities.

Because public goods and externalities are typically associated with market failures, production of them by the cooperative may require special attention in the decision-making process. Also, because the cooperative may not be able to exclude nonmembers from the benefits of public goods paid for by members of the cooperative association, members may have an incentive to become nonmembers. Therefore, the cooperative decision-maker may have to act to combat problems which result from free-riding.

Closely related to the decisions which the cooperative decision-maker must make on prices are the decisions he must make concerning the determination of patronage refunds and the allocation of joint fixed costs among goods and services. Because the cooperative cannot determine the actual costs of producing the products it sells to its patrons at the time of sale, it charges its patrons a cash price for each product at the time of sale. Then after the costs are determined, the cooperative will grant its patrons patronage refunds.

The decisions on patronage refunds are linked to those on prices. A high cash price may discourage sales but will ensure the cooperative's

ability to grant patronage refunds later. Thus, the cooperative decision-maker must decide on low prices or large patronage refunds. His decision may be influenced by tax considerations or by the fact that deferred patronage refunds provide the cooperative a useful source of capital.

One method which can be used by the cooperative in determining the per-unit patronage refund for a particular good or service consists of subtracting the average variable cost of providing the good or service and the average fixed cost allocated to it from its cash price. In this case, the method the cooperative uses to allocate joint fixed costs, as well as the method it uses to determine patronage refunds, becomes a factor in determining the effective prices which, in turn, affect the production decisions of the member patrons and, consequently, their welfare.

More often, related goods or services are grouped together in departments and a single per-unit patronage refund is determined for each department. Even in this case, it may be necessary to allocate joint fixed costs among the departments. If this is so, the cost allocations will still affect the determination of patronage refunds and, thus, the effective prices.

In the extreme, a cooperative may determine a single per-unit patronage refund for all of the goods and services which it sells. Both this and the case in which more than one product is included in a department have the potential for allowing the "netting" of losses in one line of products against the gains of others. This is in direct conflict with one of the "principles of cooperation."

The "principles of cooperation" are a set of fundamental principles to which most of those involved in the cooperative movement subscribe and by which they feel cooperatives should be run. There is no one exact set of these principles - numerous authors have offered theirs. (For example, see [1, pp. 47-70; 2, pp. 183-203; 7, p. 81; and 47, pp. 201-212].) However, certain of these principles have gained the acceptance of a majority of cooperators, and the principle of "service at cost" is one of these.

In actual practice, many cooperatives choose to ignore one or more of these principles. Although some of these principles have been incorporated into law, in many cases, the cooperative decision-maker may be in the situation where he must decide between operating in accordance to a particular principle or not. At least, he may be interested to know how practicing these principles affects his operation in terms of cost or efficiency.

Strict adherence to the principle of service at cost may restrict the ability of the multi-product cooperative to achieve its objectives. Proprietary firms that compete with cooperatives may often use loss-leaders in attracting business. A loss-leader is a product for which the price charged is lower than the average cost of providing the product. Although when taken by itself, the loss-leader may be a money-loser, it may actually be a money-maker when its complementary relationships with other products are taken into account. This is because sales of a loss-leader may increase the volumes of other products to such an extent that the firm's level of profits is higher than it would be if the price of the loss-leader were high enough to make it self-supporting.

It is possible that, in a similar manner, the cooperative may be able to further its own objectives through the proper use of loss-leaders. Thus, strict adherence to the principle of providing service at cost is at issue, and the cooperative decision-maker must decide if the cooperative will carry loss-leaders or if it will practice a policy of providing service at cost with each product line entirely supporting itself. If the cooperative decision-maker adopts a policy of loss-leaders, he must be able to determine which products the cooperative will carry as loss-leaders and what prices it will charge for them.

Other decisions and issues which concern the cooperative involve its relationship with nonmembers. The cooperative decision-maker must determine the extent to which the cooperative will do business with nonmembers and under what conditions business with nonmembers will be transacted. Nonmember purchases of supplies and services from the cooperative may be encouraged if they allow the cooperative to expand its volume to such an extent that the per-unit cost of providing goods and services to the member firms decreases. Likewise, marketing of nonmember products through the cooperative may be desirable if the increased supply of products under management of the cooperative results in a decrease in the per-unit cost of handling or processing the members' products or an increase in the bargaining power of the cooperative and, consequently, the prices it receives.

On the other hand, there may exist reasons for limiting the business between nonmembers and the cooperative. The extent to which the cooperative can do business with nonmembers may be restricted by law, or the

cooperative may be prohibited by principle from doing any business whatsoever with nonmembers. (Abrahamsen [1, pp. 64-65], for example, discusses the principle of "exclusive trading with members.")

If the cooperative does have nonmember patrons, the cooperative decision-maker must decide whether or not to grant them patronage refunds on a par with member patrons. There may be a limited tax advantage to the cooperative from doing so. On the other hand, retention of the net savings from nonmember business may provide an important source of capital for the cooperative.

Finally, the cooperative decision-maker must determine the cooperative's membership policy. Again, the cooperative may be forced to practice an open membership policy. (The principle of "open membership" is included or discussed in [1, pp. 63-64; 7, p. 189; and 47, pp. 201-202].) In other cases the cooperative decision-maker may choose to pursue an open membership policy for entirely economic reasons. An expansion of membership may increase the cooperative's bargaining power or may allow it to more economically serve its existing members. Also, if activities of the cooperative benefit nonmembers in such a way that there is incentive for members to become nonmembers, the cooperative may have to take steps which will encourage membership simply to keep from losing existing members. On the other hand, the cooperative decision-maker may choose to pursue a restricted membership policy because an expansion of membership may make its existing members worse off.

Review of the Literature[1]

In the early literature on cooperative associations, there was no mention of a cooperative decision-maker. Instead, the efforts of the early writers on cooperative associations were focused upon the entrepreneurs of the individual member firms as they were seen to be the only decision-makers within the cooperative association, utilizing the resources within their individual plants and those of the cooperative to maximize their individual profits.

The work of Emelianoff [18], has been regarded by many as one of the first important attempts at objectively analyzing the cooperative association. Emelianoff viewed the cooperative association as an organization of economic units, each maintaining its economic independence, but conducting and coordinating their business activities through an agency, owned and controlled by them. He stressed that the cooperative was only an extension of its member firms and that it was not an "enterprise," defined by Emelianoff as a profit-acquiring economic unit. Consequently, because of the absence of profit rewards, the existence of a cooperative entrepreneur or decision-maker could not be assumed.

[1]This review of literature does not include a vein of literature on cooperatives which has appeared for the most part in the American Economic Review [16, 23, 38, and 53]. McGregor [38] calls the subject of these articles "production cooperatives" and distinguishes them from what he calls "marketing cooperatives." According to him, marketing cooperatives may be involved in production in the sense that they process the products of their members, but members perform their producer role outside the cooperative in contrast to the production cooperative. Most of the models reviewed in this section are marketing cooperatives, to use McGregor's terminology. In the extreme, however, production cooperatives can be viewed as a special case of the marketing cooperative in which the cooperative purchases its members' labor and "processes" it by converting it into a finished product which it sells.

This concept of a cooperative association was shared by others, including Robotka [45] and Phillips [44, pp. 74-75], who stated that "the cooperative has no more economic life or purpose apart from that of the participating economic units than one of the individual plants of a large multi-plant firm." Phillips accepted Emelianoff's idea of a cooperative association as an organization of economic units, each maintaining its economic independence in seeking profits. The cooperative was not seen as a decision-making unit, but as a simple extension of the member firms which are the decision-making units.

In the Phillips model, the cooperating firms individually attempt to maximize their profits, and each is treated as a multi-plant, vertically integrated firm. The output of the joint plant or cooperative is assumed to be the raw product input of the individual plants of the member firms or, alternatively, the output of the individual plants is assumed to be the raw product input of the cooperative.

As a multi-plant firm, each cooperating firm must make decisions concerning the allocation of its productive resources between the cooperative and its individual plant or plants. Within this framework, Phillips attempted to outline a set of rules for the optimum behavior of a member firm, given its objective of maximizing profits. According to Phillips, a member firm maximizes its profits by equating the sum of the marginal cost in its individual plant or plants and the marginal cost in the cooperative with the marginal revenue it receives from the market in which its output is sold.

The Phillips model was criticized by Aresvik [5], who argued that the marginal cost that a member firm incurs in the cooperative is not the marginal cost of the cooperative plant but the average cost of the plant. Aresvik also argued that the marginal revenue that a member firm receives from a marketing cooperative is not the marginal revenue received by the cooperative but the average revenue received by the cooperative. Thus, according to Aresvik, a member firm participating in a marketing cooperative maximizes its profits by equating the sum of the marginal cost in its individual plant or plants and the average cost in the cooperative with the average revenue received by the cooperative in the market in which its output is sold. Aresvik did not, however, dispute Phillips' contention that it is the member firms and not the cooperative that are decision-makers. Instead, he stated that Phillips was correct in indicating that the member units, not the cooperative, are the maximizing units.

Trifon [51] indicated that neither Phillips or Aresvik was correct. He suggested that in the example of a procurement or supply cooperative, each member patron maximizes its profits by equating the sum of the marginal cost in its individual plant and the marginal cost it incurs in the cooperative with the marginal revenue it receives from the market in which its output is sold. However, he argued that by increasing its patronage, an individual patron incurred only a portion of the additional cost to the cooperative while assuming a larger share of the initial costs. Thus, the marginal cost the member patron incurs in the cooperative is neither the marginal or average cost curve of the cooperative.

He also suggested that as each individual member patron independently attempted to maximize its profits, there was no guarantee that an equilibrium would be reached.

Because many of the early writers did not perceive the existence of a cooperative decision-maker, the objectives of the cooperative received little attention from them. Many of the early writers on cooperative associations agreed that the purpose of the cooperative was to benefit its members. According to Robotka [45, pp. 97-98], most American economists who had written on the subject of cooperative associations would have accepted the idea that cooperatives were operated for the benefit of their members as patrons. Stokdyk stated that members established cooperatives as a means of increasing the profits from their individual operations [included in 2, p. 69].

In the minds of many writers, the primary means by which this purpose could be fulfilled was by the provision of services at cost. In fact, the concept of service at cost has been so widely accepted that it has been regarded, even now, as one of the "principles of cooperation" [1, pp. 54-56; 2, pp. 191-192; and 7, pp. 55-57]. According to Stokdyk [2, p. 69], the stated objective of most cooperatives was "to perform a given service or function at cost in order to increase the returns or profits of its members."

Clark [11] presented a model in which the cooperative attempted to minimize the cost of providing a given service to its member firms. In the Clark model, the cooperative either supplies goods and services to be used as inputs by the member firms, or markets their output. The

cooperative operates at the level which corresponds to the minimum average cost or the point at which average cost equals marginal cost.

Aizsilnieks [3] criticized the model, arguing that the cooperative cannot have an independent or autonomous output policy because it must provide the quantity of services that the member firms demand. Thus, according to him, it is the cost curves of the individual member firms and not those of the cooperative which determine its output level.

The belief, expressed by Aizsilnieks and other writers, that the cooperative passively meets the demands of the member firms and that no cooperative decision-maker exists was challenged by Savage [48] in a criticism of the Phillips model. Savage contended that even if the cooperative decision-maker does not seek profits for himself, he is capable of making entrepreneurial decisions that affect the environment of the cooperative's member firms. Savage argued that by reserving use of the term "firm" for describing economic units which seek profits, Phillips was ignoring the existence of cooperatives as "going concerns," as recognized by farmers and cooperative leaders.

Enke [19] presented an early model of a consumer cooperative association in which a decision-maker makes decisions concerning the operation of the cooperative. In his model, the pricing policies of the cooperative are motivated by a cooperative objective function. According to Enke, an objective of maximizing the profit of the cooperative would take the members into account as owners only and would ignore them as patrons. Similarly, an objective of minimizing the prices charged members by the cooperative would take them into account as patrons but would ignore them

as owners. The alternative he suggested, maximization of the members' net consumer surplus, would consider the members as both owners and patrons. According to Enke, the decision-maker for a cooperative which has this objective should set the price the cooperative charges its member firms for a particular product equal to the marginal cost of producing it.

Helmberger and Hoos [28] presented the first model of a producer cooperative association in which the cooperative is a decision-making unit. They contended that the member firms of a cooperative association cannot be assumed to manage the cooperative as in the Phillips model. Instead, they felt that by joining a cooperative association, a member firm commits itself to abide by group decisions. Furthermore, by assuming maximizing behavior on the part of the cooperative, they showed that behavioral relations and positions of equilibrium can be derived through marginal analysis.

In their short-run model of a cooperative marketing association, the cooperative attempts to maximize the amount available for payment to its member firms for the raw material which they choose to supply the cooperative. The cooperative combines productive services with the raw material to create a finished commodity which it markets. Treating the amount of the raw material supplied to it by the member firms as a parameter beyond its control, the cooperative maximizes the price it is able to pay member firms for the raw material by equating the price of the finished commodity to the marginal cost of producing it.

According to Helmberger and Hoos, for any given level of raw material supplied to it by its member firms, there exists a unique maximum price that the cooperative is able to pay the member firms for the raw material. They called this relationship between the quantity of raw material supplied to it and the maximum price the cooperative can pay its member firms for the raw material the short-run net returns function.

Assuming that each member firm is a price-taker with respect to the price it receives from the cooperative for the raw material which it supplies to the cooperative, Helmberger and Hoos also determined an aggregate supply function for the member firms. The point at which this short-run supply function and the short-run net returns function intersect determines the quantity of raw material the member firms supply the cooperative and the price the cooperative is able to pay them for it.

Helmberger and Hoos demonstrated that their model is consistent with Aresvik's contention that a cooperating firm maximizes its profits by equating the sum of the marginal cost in its individual plant or plants and the average cost in the cooperative with the average revenue received by the cooperative in the market in which its output is sold.

In addition to their short-run analysis, Helmberger and Hoos presented two long-run analyses of the cooperative association in which the assumption of a fixed plant is dropped. In one analysis, the cooperative maximizes the price it pays the member firms subject to the constraint that all costs are met. After determining the supply of raw material which results in the maximum price, the cooperative maintains that price by pursuing a policy of restricted membership. In the other analysis,

the cooperative pursues a policy of open membership and maximizes the price it pays its members for the raw material, subject to the constraint that the costs are met, for any amount of raw material which a freely variable number of member firms wishes to supply.

Hardie [25] attempted to generalize the Helmberger and Hoos model of a marketing cooperative by constructing a model of a multi-product marketing cooperative. Unlike that of the Helmberger and Hoos model, the production function of the cooperative in the Hardie model is assumed to be linear and homogeneous with discontinuous factor substitution so that the model can be expressed within a linear programming framework. The finished product prices are assumed to be fixed, and average variable costs are assumed to be constant. In addition, the raw materials are assumed to be supplied exclusively by member firms, and they are assumed to be the only limitational inputs.

As in the Helmberger and Hoos model, the cooperative's demand for the raw materials produced by its member firms is a derived demand, resulting from the demand which it faces for the finished commodities. The cooperative is assumed to maximize the returns to the member-supplied products. In addition, it is assumed to distribute all of the net proceeds it receives from marketing the finished products to the member firms by granting each unit of a given product the same return. Unlike the cooperative in the Helmberger and Hoos model, the cooperative in the Hardie model must determine the division of the net proceeds among the different classes of member products since there are more than one.

Hardie suggested that distribution of the net proceeds within his model be carried out on the basis of shadow prices. If each member-supplied raw material is paid its shadow price, the net proceeds are maximized and the cooperative distributes these proceeds to the member firms in such a manner that each unit of a given product receives the same return. In addition, shadow prices provide a criterion for the division of the net proceeds among the various classes of member products.

As Hardie pointed out, when the raw materials are assigned prices equal to their shadow prices, the net revenue from any unit of a finished commodity is equal to the cost of the raw materials used by the cooperative in producing it. Thus, use of shadow prices in the linear programming framework results in the equation of marginal revenues with marginal costs and of average revenues with average costs. In addition, since the shadow prices are the per-unit monetary contributions of the raw materials supplied by the member firms to the finished products of the cooperative, each member receives the portion of the net proceeds earned by his output if the raw materials are assigned prices equal to their shadow prices.

Although these properties of Hardie's simple linear model are appealing, the assumptions that must be made to obtain them are restrictive. Hardie attempted to make his model less restrictive by considering downward-sloping demand curves for some of the finished products and average variable costs which are not constant, suggesting that these could be implemented by using separable programming techniques. In

addition, he suggested that nonmember raw products could be introduced into the model by the inclusion of purchasing variables and that additional limitational inputs could be considered by the addition of more constraints. However, not all of the net proceeds are assigned to the member firms when additional constraints are added to the problem. Consequently, Hardie had to correct this by using the pooling constraint method.

Ladd [35] extended the analysis of Helmberger and Hoos in another model of a marketing cooperative. In the Ladd model, the market price of the raw product sold through the cooperative is not fixed as in the Helmberger and Hoos model but is a variable dependent upon the cooperatives actions. Also, Ladd presented rules for determining the optimum membership of the cooperative association and provided instruments which the cooperative could use to reach it. In addition, Ladd's cooperative provides three services--an item sold both to members and nonmembers which is used as a productive input by them, a service provided free of charge to members, and a bargaining service which benefits both members and nonmembers by affecting the price they receive for their raw product.

Ladd alternatively considered the Helmberger and Hoos objective of maximizing the raw material price received by its members and the objective of maximizing the quantity of the raw material marketed through the cooperative. There are three instruments which are available to the cooperative for attaining these two objectives. These instruments are the price charged by the cooperative for the service used as a productive input by members and nonmembers, the quantity of the excludable public

good provided to the members of the cooperative, and the quantity of the bargaining service performed by the cooperative. Using these instruments, Ladd derived the first-order maximization conditions for each of the two objectives and showed that they are substantially different from each other and from those of a profit-maximizing proprietary firm.

As did Hardie, Bar [8] used a linear programming framework in presenting a short-run model of the cooperative association. In this model, the cooperative provides a number of services which the member firms use in their productive processes. These may include marketing services. Aware that both the member firms and the cooperative represent decision-making units, Bar gave each an optimizing role in his model. Each member firm attempts to maximize its surplus of income over costs, which include payments to the cooperative for the services which it provides. The cooperative itself attempts to maximize the aggregate surplus of its member firms.

It is assumed that the variable costs of the services provided by the cooperative are charged to the members per unit of service used. Fixed costs are also assumed to be charged to member firms in proportion to the quantity of services used. Thus, the problem facing the cooperative is one of determining the optimal per-unit charges for covering fixed costs.

Through use of linear programming theory, Bar demonstrated that the cooperative must set a per-unit charge for fixed costs for each service equal to the shadow price of the resource used to provide the service if it is to achieve its objective of maximizing aggregate surplus. Bar

noted, however, that not all fixed costs are necessarily covered through use of this pricing policy. Hence, he suggested that the cooperative must set its per-unit charges for fixed costs so as to cover all fixed costs with as little deviation from the optimal per unit charges as possible.

Problem Selection

The existing literature on producers cooperative associations is deficient in several respects. First of all, several of the models presented in the literature fail to recognize the existence of a cooperative decision-maker. Only the more recent models of Helmberger and Hoos, Hardie, Ladd, and Bar include cooperative decision-makers and give them maximizing roles.

Second, none of the models presented in the literature explore the pricing decisions which must be made by the multi-product cooperative. The cooperatives in the models of Phillips, Clark, and Helmberger and Hoos are all single-product cooperatives. Although the cooperative in the Ladd model provides three services to member patrons, it is not a multi-product cooperative in the sense that it sells more than one input to patrons or purchases more than one output from them. The cooperative in both the Hardie and Bar models are multi-product cooperatives in this sense. However, the cooperative in the Hardie model is strictly a marketing cooperative--it provides no services to its patrons. Likewise, although the services provided by the cooperative in the Bar model may include marketing services, they are not explicitly analyzed.

In any case, none of the models which contain multi-product cooperatives are used to consider complementarity and substitution between products in the pricing of services. Instead, it is implicitly assumed in every model that patrons are charged a price for each service equal to the average cost of producing it.

Third, none of the models of cooperative associations make any mention of patronage refunds although they are an integral and important part of real-world cooperatives. Implicitly, it is assumed that the cooperative has perfect knowledge of what its costs are so that it can determine prices in such a manner that there is no surplus to be distributed at the end of the accounting period. In actuality, cooperatives seldom if ever can be certain of what their costs are until the end of the accounting period. Further, patrons do not know what their patronage refunds will be at the time they make their production decisions, but must wait until the end of the accounting period to see. No model in the literature suggests a mechanism for incorporating the expected patronage refunds of a patron into his production decisions.

Fourth, only the Bar model discusses fixed costs at all, and it does not analyze the problem of allocating joint fixed costs among services. Instead, the method of charging overhead costs to the accounts of the various services is given rather than determined within the framework of the model. Further, the Bar model is used only in determining the per-unit charges for fixed costs for each service, and the method employed is not entirely satisfactory.

Fifth, the subjects of public goods and externalities have also been largely overlooked by the literature on cooperative associations. Only the Ladd model considers the provision of a public good by the cooperative. The public good in the Ladd model is one which affects price. There is no discussion of public goods which affect production or of externalities in any of the models in the literature.

Sixth, in all but the Ladd model, member patrons are assumed to deal exclusively with the cooperative. For example, in the Helmberger and Hoos model and in the Hardie model, the member patrons are contractually bound to do so. In the Bar model, they are obliged to do so according to the principles of cooperation. A general model would not ignore the possibility of members dealing with organizations outside the cooperative association.

Finally, only the Ladd model considers trading between the cooperative and nonmembers. Again, a general model would not ignore this possibility.

In view of the problems of the cooperative association which have not been dealt with adequately in the literature, an attempt is made in this study to develop a more general short-run model. Both the cooperative and the patrons in this model are multi-product organizations. The cooperative both markets outputs of those it serves and provides them with factors of production.

Some of these factors of production are public goods. These public goods affect production but not prices. The complexities of including public goods which affect prices are avoided in this study. Similarly,

the discussion of externalities between firms is left to the public finance literature.

Within this model, it is assumed that the cooperative does not know its costs until the end of its accounting period. Thus, the determination of patronage refunds is necessary. Further, it is assumed that patrons do not know what their patronage refunds are until the end of the accounting period and that the allocation of joint fixed costs is not given but determined within the model.

Finally, it is assumed that member firms do not need to deal exclusively with the cooperative and that nonmembers, as well as members, trade with the cooperative.

The model of the cooperative and the models of the member and nonmember patrons presented in this study are nonlinear programming models. Nonlinear programming is used so that the model can be as general as possible and so that the results of the model are not contingent upon the assumptions of more specific programming tools. This is a normative-prescriptive study in that it attempts to explain how cooperatives should behave if they are to act to achieve specified goals, not how they actually behave.

Following Chapters

In Chapter II, the models of a typical member patron and of a typical nonmember patron are presented.

In Chapter III, a model of the cooperative in which the total profits of the member patrons are maximized is presented. In this chapter, the

Kuhn-Tucker conditions corresponding to the model are presented, but are not interpreted.

In Chapter IV, the Kuhn-Tucker conditions presented in Chapter III are interpreted. Several simplifications of the model are presented and are compared to the models presented in the literature.

In Chapter V, the general model of the cooperative is extended to include consideration of the future effects on the member patron's profits of current decisions.

Finally, Chapter VI consists of a summary, conclusions, and suggestions for further research.

CHAPTER II. PATRON MODELS

General Model

In this chapter, a model of a typical member patron and a model of a typical nonmember patron are presented. These models are sub-models of the general model of the cooperative association. In the general model of the cooperative association, it is assumed that there are more than one member patron and more than one nonmember patron as well as the cooperative.

The relationships between the various sub-models are indicated in Figure 2.1. In this figure, flows of goods are indicated by heavy arrows while lighter arrows are used to indicate flows of cash or credit. Broken arrows are used to indicate flows of patronage refunds.

As can be seen from the figure, the cooperative purchases unprocessed products (set X) from member and nonmember patrons and supplies them with variable inputs (set Y) which they use in production. The cooperative determines the price it will offer its patrons for each of the unprocessed products it purchases from them. Similarly, it determines the price it will charge its patrons for each of the variable inputs which it sells them.

Not all of the inputs which the cooperative supplies to its patrons are sold, however. Some of them are public goods (set G). Because it is assumed that nonpayers cannot be excluded from enjoying the benefits of these public goods, the cooperative does not sell them. Instead, it

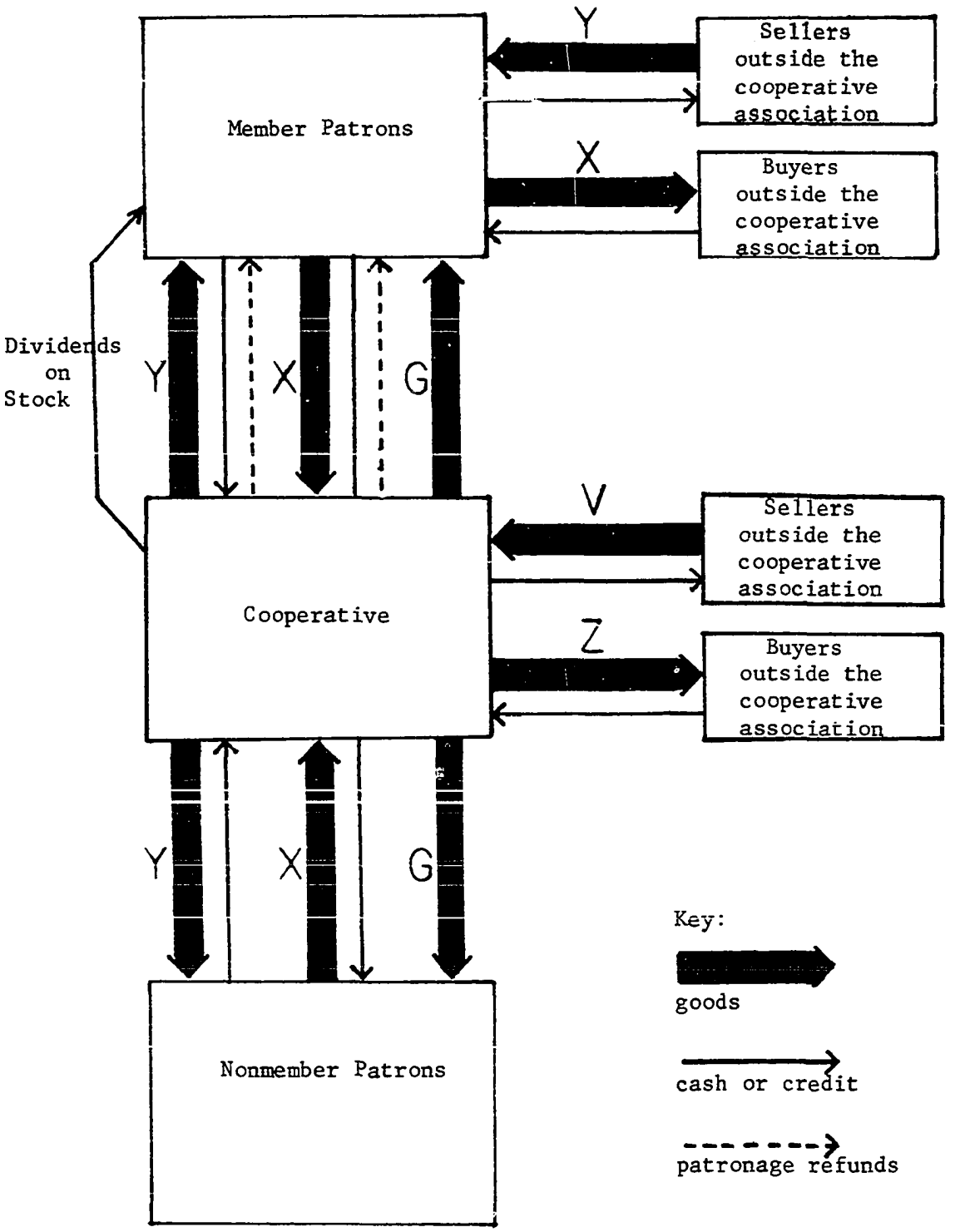

Figure 2.1. Model of the cooperative association.

provides them to both member and nonmember patrons free of charge and finances them from other business.

In addition to doing business with member and nonmember patrons, the cooperative deals with buyers and sellers outside the cooperative association. The cooperative purchases variable inputs (set V) for use in processing the unprocessed products which it purchases from its patrons and for use in producing the inputs which it supplies its patrons. It also sells the processed products (set Z).

In this model, a product of the member and nonmember firms which is simply marketed by the cooperative could be included as a special case of an unprocessed product purchased by the cooperative and sold as a processed product without the use of inputs. However, because it is assumed that the marketing of any product through the cooperative requires the use of some inputs, there is technically no difference between a product processed by the cooperative and one marketed through it and no distinction is made between the two. Similarly, no distinction is made between a variable input supplied to member and nonmember firms by the cooperative which is produced by the cooperative and one which is simply purchased by the cooperative and resold to member and nonmember firms.

The cooperative distributes patronage refunds to its member patrons, but it is assumed that there is no legal or economic reason for it to distribute them to nonmembers. The cooperative also pays members dividends on stock, but it is assumed for convenience, that nonmembers hold no stock in the cooperative.

Finally, it can be observed from the figure that member patrons do not do business exclusively with the cooperative. Instead, they purchase variable inputs from outside the cooperative association as well as from the cooperative. The relationships between nonmember patrons and outside markets are not represented in the figure because they are not relevant to the model.

Model of a Member Patron

In the sub-models of the typical member patron and the typical nonmember patron, it is assumed that each firm attempts to maximize its profit. This is the assumption common to all of the short-run models of cooperative associations reviewed in this study.

The set of products produced by the member and nonmember patrons is represented by X. The subset of products in X which are sold to the cooperative is represented by X_c while the subset of products which are sold to buyers outside the cooperative association is represented by X_o. Similarly, the set of variable factors of production purchased by the patron firms is represented by Y. The subset of variable inputs in Y purchased from the cooperative is represented by Y_c while the subset of variable inputs purchased from sellers outside the cooperative association is represented by Y_o. Both the member and nonmember patrons are assumed to be price-takers with respect to all of the prices they pay for variable inputs and receive for products.

The set of fixed factors of production which are available to the typical member patron or to the typical nonmember patron is represented

by W_f. Just as each of the products in set X and each of the variable factors in set Y have prices associated with them, the fixed factors in set W_f have per-unit costs attached to them.

In this model, it is assumed that the patrons have perfect knowledge of the prices. In other words, they are assumed to have full knowledge of all prices at the time they make their production decisions. However, it is assumed that the cooperative determines the per-unit patronage refunds for the products which it buys and the inputs which it sells at the end of its accounting period, after all purchases and sales have been made. Thus, the member patrons have only a limited knowledge, based on past refunds, of what the per-unit patronage refunds will be at the time they make their production decisions.

Hence, the typical member patron attempts to maximize its profit by maximizing its expected profit:

$$\pi = \sum_{i \in X} p_i q_i - \sum_{i \in Y} p_i q_i - fc + ds + pvpr \qquad (2.1)$$

where p_i and q_i are, respectively, the price paid or received and the quantity of the i-th product or factor, where fc is the fixed costs of the firm, where ds is the dividends on stock held by the member patron, and where pvpr is the present value of the patronage refunds which the firm expects to be allocated. The latter can be expressed:

$$pvpr = \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] \sum_{i \in C} r_i^* q_i \qquad (2.2)$$

where s is the proportion of patronage refunds paid in cash and (1-s) is the proportion deferred to a revolving fund of length τ. The symbol C

represents the set of products sold to and factors purchased from the cooperative and the symbol r_i^* represents the expected per-unit patronage refund on the i-th product or factor in this set. The symbol d represents the discount rate. In this problem, the appropriate discount rate might be the opportunity cost represented by the interest rate paid by the patron on long-term debt.

It is assumed that the expected per-unit patronage refund on the i-th product or factor is a function of the actual per-unit patronage refunds on the same product or factors in past periods:

$$r_i^* = r_i^*(r_i(t-1), r_i(t-2), \ldots) \text{ for all } i \in C. \tag{2.3}$$

No attempt is made within this study to specify the structure of this function. Many suitable models for specifying an expected price as a function of past prices exist elsewhere.

The technology of the firm is represented by a production function which, in its implicit form, is written:

$$\phi(q_X, q_Y, q_{W_f}, Q_G) = 0 \tag{2.4}$$

where q_X is a vector of the quantities of each of the products in set X produced by the firm, q_Y is a vector of the quantities of each of the variable factors in set Y used in production, and q_{W_f} is a vector of the quantities of each of the fixed factors in set W_f used in production. The symbol Q_G represents a vector of the quantities of each of the set G of public goods provided by the cooperative.

It is assumed that the production function 2.4 possesses continuous first- and second-order partial derivatives which are different from zero for all its nontrivial solutions and that it is written in such a way that the partial derivatives with respect to the outputs are positive and the partial derivatives with respect to the inputs are negative. It is further assumed that 2.4 is subject to diminishing returns such that all one-output production functions obtained from 2.4 by fixing the values of all other outputs are strictly concave.

The problem of the member patron is that of choosing the level of output for each product in set X, the level of each variable factor in set Y to be used in production of each product in set X, and the level of each fixed factor in set W_f to be used in production of each product in set X such that profit 2.1 is maximized. This maximization is subject, of course, to the production function 2.4 and a set of constraints which ensure use of each fixed factor does not exceed the quantity available. If the quantity of the i-th fixed factor used in production is represented by q_i and the stock of the factor is represented by q_{i0}, a constraint of this type can be expressed:

$$q_i \leq q_{i0}. \tag{2.5}$$

The Lagrangian functior for this problem can, therefore, be written:

$$\Lambda = \sum_{i\in X} p_i q_i - \sum_{i\in Y} p_i q_i + \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] \sum_{i\in C} r_i^{*} q_i$$
$$+ \Psi_1 \cdot \phi(q_X, q_Y, q_{W_f}, Q_G) + \sum_{i\in W_f} \Psi_{2i}(q_{i0} - q_i) \tag{2.6}$$

where Ψ_1 is the Lagrange multiplier corresponding to the production function 2.4 and the Ψ_{2i} are the Lagrange multipliers corresponding to the fixed-factor constraints 2.5.

Corresponding to the Lagrangian function 2.6 is a set of Kuhn-Tucker conditions. These are necessary conditions for a global maximum. They are sufficient conditions for a global maximum if the objective function is concave, the constraints are concave, and the set of feasible solutions is bounded and nonempty. It is assumed that marginal costs may increase or decrease with increases in output, but that if marginal costs are decreasing, the absolute value of the rate of decrease must be less than or equal to that of the rate of decrease of the marginal revenue function. Thus, the profit function 2.1 is concave. It has already been assumed that the production function 2.4 is concave. The fixed-factor constraints 2.5 are linear and, therefore, can be considered as concave. Thus, if it is assumed that the set of feasible solutions are bounded and nonempty, the Kuhn-Tucker conditions are necessary and sufficient for a global maximum.

The Kuhn-Tucker conditions for the problem represented by 2.6 are as follows:

for all $i \epsilon X_c$:

$$\frac{\partial \Lambda}{\partial q_i} = p_i + \left[s + \frac{(1-s)}{(1+d)^T}\right] r_i^* + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \qquad (2.7a)$$

$$\frac{\partial \Lambda}{\partial q_i} \cdot q_i = 0 \qquad (2.7b)$$

$$q_i \geq 0 \qquad (2.7c)$$

for all $i \in X_o$:

$$\frac{\partial \Lambda}{\partial q_i} = p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \tag{2.8a}$$

$$\frac{\partial \Lambda}{\partial q_i} \cdot q_i = 0 \tag{2.8b}$$

$$q_i \geq 0 \tag{2.8c}$$

for all $i \in Y_c$:

$$\frac{\partial \Lambda}{\partial q_i} = -p_i + \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] r_i^* + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \tag{2.9a}$$

$$\frac{\partial \Lambda}{\partial q_i} \cdot q_i = 0 \tag{2.9b}$$

$$q_i \geq 0 \tag{2.9c}$$

for all $i \in Y_o$:

$$\frac{\partial \Lambda}{\partial q_i} = -p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \tag{2.10a}$$

$$\frac{\partial \Lambda}{\partial q_i} \cdot q_i = 0 \tag{2.10b}$$

$$q_i \geq 0 \tag{2.10c}$$

for all $i \in W_f$:

$$\frac{\partial \Lambda}{\partial q_i} = \Psi_1 \frac{\partial \phi}{\partial q_i} - \Psi_{2i} \leq 0 \tag{2.11a}$$

$$\frac{\partial \Lambda}{\partial q_i} \cdot q_i = 0 \tag{2.11b}$$

$$q_i \geq 0 \tag{2.11c}$$

for Ψ_1:

$$\frac{\partial \Lambda}{\partial \Psi_1} = \phi(q_X, q_Y, q_{W_f}, Q_G) = 0 \tag{2.12}$$

for Ψ_{2i}, $i \epsilon W_f$:

$$\frac{\partial \Lambda}{\partial \Psi_{2i}} = q_{i0} - q_i \geq 0 \tag{2.13a}$$

$$\frac{\partial \Lambda}{\partial \Psi_{2i}} \cdot \Psi_{2i} = 0 \tag{2.13b}$$

$$\Psi_{2i} \geq 0. \tag{2.13c}$$

The interpretation of the Kuhn-Tucker conditions for this problem is facilitated by first discussing the meaning of the Lagrange multipliers. In general, the value of a Lagrange multiplier at a solution indicates how much the value of the objective function will change given a one-unit change in the corresponding constraint constant. Thus, Ψ_{2i} can be interpreted as the imputed value or shadow price of the i-th fixed factor

$$\Psi_{2i} = \frac{\partial \pi}{\partial q_{i0}} \qquad i \epsilon W_f . \tag{2.14}$$

In interpreting Ψ_1, Naylor [40, p. 328] suggests that ϕ be treated as if it were an arbitrary product. Then Ψ_1 can be interpreted as the imputed value or shadow price of ϕ. This interpretation then becomes useful in interpreting two other terms which include Ψ_1. If i is a product, the negative of the partial derivative $\partial\phi/\partial q_i$ can be interpreted as the rate of product transformation or the marginal cost of

product i in terms of ϕ. Thus, $-\Psi_1(\partial\phi/\partial q_i)$ can be interpreted as the marginal imputed cost of producing the i-th product. On the other hand, if i is an input, the partial derivative $\partial\phi/\partial q_i$ is the marginal product of input i with respect to ϕ and $\Psi_1(\partial\phi/\partial q_i)$ can be interpreted as the marginal value product of the i-th input.

Unfortunately, there are some methodological difficulties involved in using this approach,[1] and it is useful to avoid the problem of giving Ψ_1 an economic interpretation and to interpret $-\Psi_1(\partial\phi/\partial q_i)$ for outputs and $\Psi_1(\partial\phi/\partial q_i)$ for inputs directly instead of in parts. It is possible to mathematically demonstrate that at a profit-maximizing level, the former is equal to the marginal cost of producing the i-th product and that the latter is equal to the marginal value product of the i-th input without interpreting Ψ_1 if certain conditions are met.[2]

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[1]For example, because the production function is expressed in implicit form, ϕ is equal to zero. Therefore, $\partial\pi/\partial\phi$ is equivalent to $\partial\pi/\partial 0$. This term has no economic meaning.

Further, by the implicit function rule of calculus, the partial derivative of ϕ with respect to the quantity of the i-th input or output is equal to:

$$\frac{\partial\phi}{\partial q_i} = -\frac{\frac{\partial\phi}{\partial q_i}}{\frac{\partial\phi}{\partial\phi}} \quad . \tag{2.15}$$

The denominator, the partial derivative of ϕ with respect to itself, is equal to one. Thus, $\partial\phi/\partial q_i$ is equal to the negative of itself. The only way that this can be true is if $\partial\phi/\partial q_i$ is equal to zero. If this is true, interpreting $-\Psi_1(\partial\phi/\partial q_i)$ for the i-th output as its marginal imputed cost and $\Psi_1(\partial\phi/\partial q_i)$ for the i-th input as its marginal value product is meaningless.

[2]See the first two proofs in Appendix B.

It is important to point out that the marginal costs in this model are distinct from the marginal variable costs found in models of the single-product firm. Any marginal cost in this model may include the "marginal opportunity cost" of using fixed factors of production. Inclusion of this is necessary when the production of an additional unit of a particular product draws use of a fully-employed fixed factor away from the production of other products. The opportunity cost of the fully-employed fixed factor is represented by its shadow price.[1] Although this is an internal cost, it is just as real as any other. This concept of a marginal opportunity cost is recognized by Swenson [50, pp. 57-58 and p. 77] and is dealt with in greater detail by Rothman [46, pp. 58-67].

With this established, it is possible to proceed with interpreting the first set of Kuhn-Tucker conditions. These conditions are represented by 2.7a through 2.7c and correspond to the products produced by the firm and sold to the cooperative. Condition 2.7a can be rewritten:

$$p_i + \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] r_i^{*} \leq - \Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \in X_c. \tag{2.16}$$

The term on the left-hand side of the inequality is the effective price the member patron expects from the sale of product i to the cooperative and is equivalent to the cash price plus the discounted expected

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[1] These costs are considered in the total differential of the cost equation in Proof 1 through the inclusion of the term $\sum_{i \in W_f} \Psi_{2i}\, dq_i$ in A.2.

per-unit patronage refund on the product. The term on the right-hand side is equal to the marginal cost of producing product i.

Condition 2.7c requires that the quantity of product i produced must be nonnegative. If a positive quantity of product i is produced, condition 2.7b guarantees that 2.16 is an equality. In other words, for maximum profit, if the i-th product is produced, it should be produced up to the point at which the marginal cost of producing it is equal to its effective price, i.e., the cash price plus the discounted expected per-unit patronage refund on the product. If 2.16 is satisfied as a strict inequality, i.e., if the effective price of product i is less than the marginal cost of producing it, condition 2.7b guarantees that none is produced.

The next set of conditions is represented by 2.8a through 2.8c and corresponds to the products produced by the firm and sold outside the cooperative association. Condition 2.8a can be rewritten:

$$p_i \leq - \Psi_1 \frac{\partial \phi}{\partial q_i} \qquad \text{for all } i \in \bar{X}_o. \tag{2.17}$$

The term on the left-hand side of the inequality is the price of product i. Again, the term on the right-hand side is equal to the marginal cost of producing the i-th product.

Condition 2.8c requires that the quantity of product i produced must be nonnegative. If a positive quantity of product i is produced, condition 2.8b guarantees that 2.17 is an equality. In other words, for a maximum, if the i-th product is produced, it should be produced up to the point at which the marginal cost of producing it is equal to the

price. If 2.17 is satisfied as a strict inequality, i.e., if the price of product i is less than the marginal cost of producing it, condition 2.8b guarantees that none is produced.

Thus, the interpretations of conditions 2.7a through 2.7c and of 2.8a through 2.8c are very similar. The only difference between 2.16 and 2.17 is that there is no discounted expected per-unit patronage refund term in the latter since the member patron receives no patronage refunds on products sold outside the cooperative association.

The conditions represented by 2.9a through 2.9c correspond to the use of the i-th variable factor of production purchased from the cooperative. Condition 2.9a can be rewritten:

$$p_i - \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] r_i^* \geq \Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \varepsilon Y_c. \tag{2.18}$$

The term on the left-hand side of the inequality is the effective price the member patron expects to pay the cooperative for input i and is equivalent to the cash price less the discounted expected per-unit patronage refund on the input. The term on the right-hand side is equal to the marginal value product on input i.

Condition 2.9c requires that the quantity of input i used must be nonnegative. If a positive quantity of input i is used, condition 2.9b guarantees that 2.18 is an equality. In other words, for a maximum, if the i-th input is used, it should be used up to the point at which its marginal value product is equal to its effective price, i.e., the cash price less the discounted expected per-unit patronage refund on the input. If 2.18 is satisfied as a strict inequality, i.e., if the

effective price of input i is greater than its marginal value product, condition 2.9b guarantees that none is used.

Similar conditions corresponding to the use of the i-th variable input purchased from outside the cooperative association are represented by 2.10a through 2.10c. Condition 2.10a can be rewritten:

$$p_i \geq \Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \epsilon Y_o. \tag{2.19}$$

The term on the left-hand side of the inequality is the price of input i. Again, the term on the right-hand side is equal to the marginal value product of input i.

Condition 2.10c requires that the quantity of input i used must be nonnegative. If a positive quantity of input i is used, condition 2.10b guarantees that 2.19 is an equality. In other words, for a maximum, if the i-th input is used, it should be used up to the point at which its marginal value product is equal to its price. If 2.19 is satisfied as a strict inequality, i.e., if the price of input i is greater than its marginal value product, condition 2.10b guarantees that none is used.

Thus, the interpretations of conditions 2.9a through 2.9c and of 2.10a through 2.10c are also very similar. The only difference between 2.18 and 2.19 is that there is no discounted expected per-unit patronage refund term in the latter since the member patron receives no patronage refunds on inputs purchased from outside the cooperative association.

The conditions represented by 2.11a through 2.11c correspond to the use of the fixed factors of production. Condition 2.11a can be rewritten:

$$\Psi_{2i} \geq \Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \epsilon W_f. \tag{2.20}$$

As noted, the Lagrange multiplier on the left-hand side of the inequality is the imputed value or shadow price of the i-th fixed factor. The term on the right-hand side is its marginal value product.

Condition 2.11c requires that the quantity of the i-th fixed factor used must be nonnegative. If a positive quantity of the i-th fixed factor is used, condition 2.11b guarantees that 2.20 is an equality. In other words, at a maximum, if the i-th input is used, its imputed value is equal to its marginal value product. If 2.20 is satisfied as a strict inequality, i.e., if the imputed value of the i-th fixed factor is greater than its marginal value product, condition 2.11b guarantees that none is used.

This result by itself may not appear to be too meaningful. However, it assumes more meaning in the discussion, found later in this section, of competing uses for a fixed factor. It also becomes more meaningful when related to conditions 2.13a through 2.13c.

Condition 2.12 is simply a restatement of the firm's production function.

Conditions 2.13a through 2.13c correspond to the fixed-factor constraints 2.5. In fact, condition 2.13a is a restatement of 2.5. Condition 2.13c requires that the imputed value of the i-th fixed factor must be nonnegative. If the imputed value of the factor is positive, condition 2.13b guarantees that 2.13a is an equality. If 2.13a is

satisfied as a strict inequality, 2.13b guarantees that the imputed value is equal to zero.

These results can be represented by the complementary slackness conditions:

$$q_{i0} = q_i \quad \text{if} \quad \Psi_{2i} > 0, \tag{2.21a}$$

$$\Psi_{2i} = 0 \quad \text{if} \quad q_i < q_{i0} \quad \text{for all } i \in W_f. \tag{2.21b}$$

In other words, at a maximum, if the imputed value of the i-th fixed factor is positive, the stock of the factor must be exhausted. If the stock is not exhausted, i.e., if there is slack, the imputed value of the factor must be zero.

The Kuhn-Tucker conditions 2.11a through 2.11c guarantee that, at a maximum, if the i-th fixed input is used, its imputed value is equal to its marginal value product. Thus, conditions 2.21a and 2.21b imply that if the i-th fixed input is used, it will be exhausted unless its marginal value product is equal to zero. This is illustrated by Figure 2.2. If the stock of the i-th factor is equal to q_{i0}', it will be exhausted because the marginal value product or imputed value is positive at that point. On the other hand, if the stock is equal to q_{i0}'', it will only be used up to the point at which its marginal value product or imputed value is equal to zero. For the rest of this chapter, it is assumed that each of the fixed factors is used and that the marginal value product of each is positive so that the factor is exhausted. This does not seem to be an unrealistic assumption.

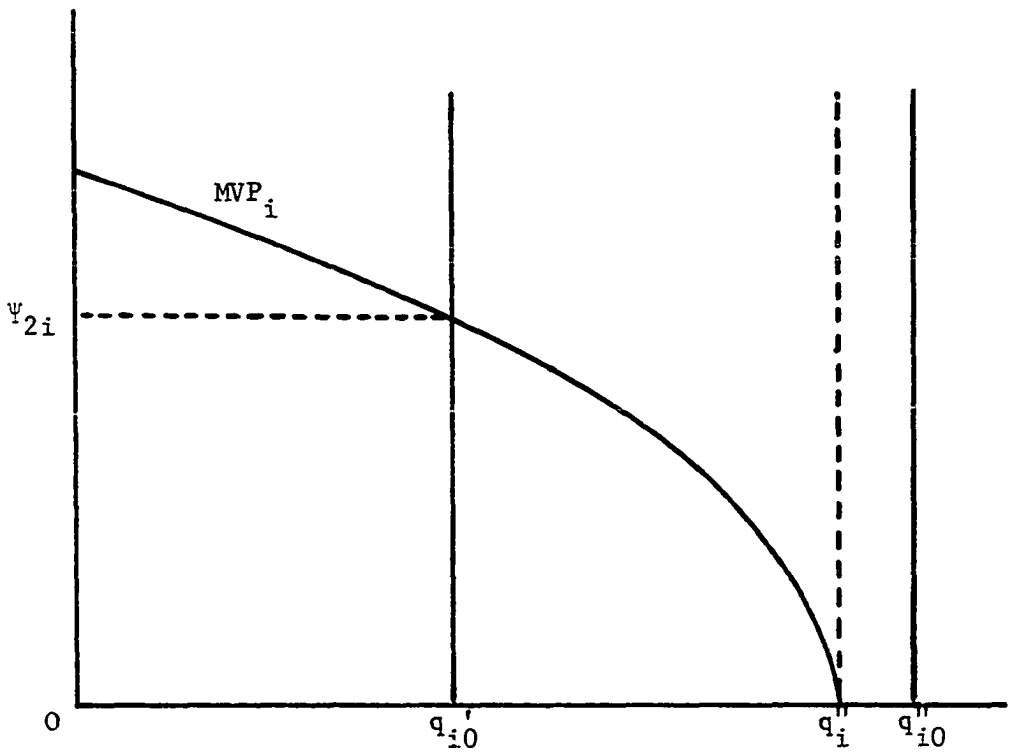

Figure 2.2. Use of a fixed factor.

Letting:

$$p_i^* = p_i, \text{ the cash price, for all } i \epsilon X_o, Y_o,$$

$$= p_i + s \cdot r_i^* + (1-s)\, r_i^*/(1+d)^T, \text{ the effective price,}$$

$$\text{for all } i \epsilon X_c,$$

$$= p_i - s \cdot r_i^* - (1-s)\, r_i^*/(1+d)^T, \text{ the effective price,}$$

$$\text{for all } i \epsilon Y_c,$$

$$= \Psi_{2i}, \text{ the imputed value or shadow price, for all } i \epsilon W_f,$$

additional light can be shed on the Kuhn-Tucker conditions. If all q_i exceed zero, conditions 2.16 through 2.20 are equivalent to:

$$p_i^* = -\Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \epsilon X \tag{2.22}$$

and:

$$p_i^* = \Psi_1 \frac{\partial \phi}{\partial q_i} \quad \text{for all } i \epsilon Y, W_f. \tag{2.23}$$

For an output, condition 2.22 requires that it be produced up to the point at which its marginal cost equals its price or effective price. For an input, condition 2.23 requires that it be used up to the point at which its marginal value product equals its price, effective price, or imputed value.

Selecting any two of the equations of the type 2.23, dividing one by the other, and using the implicit function rule results in the condition:

$$\frac{p_k^*}{p_\ell^*} = \frac{\frac{\partial \phi}{\partial q_k}}{\frac{\partial \phi}{\partial q_\ell}} = -\frac{\partial q_\ell}{\partial q_k} \quad \text{for all } k, \ell \epsilon X. \tag{2.24}$$

It requires that the marginal rate of transformation for every pair of outputs, holding the levels of all other outputs and all inputs constant, must equal the ratio of their prices or effective prices. A similar condition for all k, $\ell \epsilon Y$, W_f can be derived by selecting any two of the equations of the type 2.23. It requires that the marginal rate of technical substitution for every pair of inputs, holding the levels of all outputs and all other inputs constant, must equal the ratio of their prices, effective prices, or imputed values.

Selecting any one of the equations of the type 2.22 and any one of the equations of the type 2.23, dividing the latter by the former, and using the implicit function rule results in:

$$\frac{p_k^*}{p_\ell^*} = \frac{\frac{\partial \phi}{\partial q_k}}{\frac{\partial \phi}{\partial q}} = \frac{\partial q_\ell}{\partial q_k} \qquad \text{for all } k \epsilon Y, W_f; \text{ all } \ell \epsilon X. \tag{2.25}$$

Multiplying 2.25 by p_ℓ^* results in the condition:

$$p_k^* = p_\ell^* \cdot \frac{\partial q_\ell}{\partial q_k} \qquad \text{for all } k \epsilon Y, W_f; \text{ all } \ell \epsilon X. \tag{2.26}$$

This requires that the marginal value product of each input with respect to each output must equal the price, effective price, or imputed value of the input. It follows that the marginal value products of an input with respect to all outputs must be equal. Given conditions 2.11a through 2.11c, it also follows that if the marginal value product of a fixed input used in a particular output is less than the imputed value

of the input determined by its marginal value product in other uses, it will not be used in the production of the output.

Model of a Nonmember Patron

Using the notation developed in the previous section, it is possible to describe a model of a typical nonmember patron. The profit function of the nonmember patron is similar to that of the typical member patron except that it does not include a dividend-on-stock term or a present-value-of-patronage-refunds term:

$$\pi = \sum_{i \in X} p_i q_i - \sum_{i \in Y} p_i q_i - fc. \tag{2.27}$$

The corresponding Lagrangian function is:

$$\Delta = \sum_{i \in X} p_i q_i - \sum_{i \in Y} p_i q_i + \Psi_1 \cdot \phi(q_X, q_Y, q_{W_f}, Q_G) + \sum_{i \in W_f} \Psi_{2i}(q_{i0} - q_i) \tag{2.28}$$

and the corresponding Kuhn-Tucker conditions are:

for all $i \in X$:

$$\frac{\partial \Delta}{\partial q_i} = p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \tag{2.29a}$$

$$\frac{\partial \Delta}{\partial q_i} \cdot q_i = 0 \tag{2.29b}$$

$$q_i \geq 0 \tag{2.29c}$$

for all $i \in Y$:

$$\frac{\partial \Delta}{\partial q_i} = -p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} \leq 0 \tag{2.30a}$$

$$\frac{\partial \Delta}{\partial q_i} \cdot q_i = 0 \tag{2.30b}$$

$$q_i \geq 0 \tag{2.30c}$$

for all $i \in W_f$:

$$\frac{\partial \Delta}{\partial q_i} = \Psi_1 \frac{\partial \phi}{\partial q_i} - \Psi_{2i} \leq 0 \tag{2.31a}$$

$$\frac{\partial \Delta}{\partial q_i} \cdot q_i = 0 \tag{2.31b}$$

$$q_i \geq 0 \tag{2.31c}$$

for Ψ_1:

$$\frac{\partial \Delta}{\partial \Psi_1} = \phi(q_X, q_Y, q_{W_f}, Q_G) = 0 \tag{2.32}$$

for all $i \in W_f$:

$$\frac{\partial \Delta}{\partial \Psi_{2i}} = q_{i0} - q_i \geq 0 \tag{2.33a}$$

$$\frac{\partial \Delta}{\partial \Psi_{2i}} \cdot \Psi_{2i} = 0 \tag{2.33b}$$

$$\Psi_{2i} \geq 0. \tag{2.33c}$$

The interpretation of these conditions is similar to that of those of the model of the typical member patron.

Output Supply and Input Demand Functions

Output supply functions and input demand functions for the typical member and the typical nonmember patrons can be derived from the Kuhn-Tucker conditions. For example, if it is assumed that the typical member patron produces all of the products in set X, uses all of the variable factors in set Y, and exhausts all of the fixed factors in set W_f, conditions 2.16 through 2.20, as well as 2.13a, can be written as equalities. Thus, the Kuhn-Tucker conditions are equivalent to:

$$p_i + \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] r_i^* + \Psi_1 \frac{\partial \phi}{\partial q_i} = 0 \quad \text{for all } i \epsilon X_c \tag{2.34}$$

$$p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} = 0 \quad \text{for all } i \epsilon X_o \tag{2.35}$$

$$p_i + \left[s + \frac{(1-s)}{(1+d)^{\tau}}\right] r_i^* + \Psi_1 \frac{\partial \phi}{\partial q_i} = 0 \quad \text{for all } i \epsilon Y_c \tag{2.36}$$

$$p_i + \Psi_1 \frac{\partial \phi}{\partial q_i} = 0 \quad \text{for all } i \epsilon Y_o \tag{2.37}$$

$$\Psi_1 \frac{\partial \phi}{\partial q_i} - \Psi_{2i} = 0 \quad \text{for all } i \epsilon W_f \tag{2.38}$$

$$\phi(q_X, q_Y, q_{W_f}, Q_G) = 0 \tag{2.39}$$

$$q_{i0} - q_i = 0 \quad \text{for all } i \epsilon W_f. \tag{2.40}$$

These conditions are the first-order conditions for the classical programming problem of maximizing the Lagrangian function 2.6. They can be solved for the optimal values of the variables as functions of the parameters if the determinant of the Jacobian matrix J is nonvanishing, where:

$$
J = \left[\begin{array}{cccc|c|cccc}
\Psi_1\phi_{11} & \Psi_1\phi_{12} & \cdots & \Psi_1\phi_{1j} & \phi_1 & 0 & 0 & \cdots & 0 \\
\Psi_1\phi_{21} & \Psi_1\phi_{22} & \cdots & \Psi_1\phi_{2j} & \phi_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\Psi_1\phi_{i1} & \Psi_1\phi_{i2} & \cdots & \Psi_1\phi_{ij} & \phi_i & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 & & & & & 0 & 0 & \cdots & 0 \\
\hline
\phi_1 & \phi_2 & \cdots & \phi_j & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0
\end{array}\right] \qquad (2.41)
$$

and where ϕ_i represents the first-order partial derivative $\partial\phi/\partial q_i$ and ϕ_{ij} represents the second-order partial derivative $\partial^2\phi/\partial q_i \partial q_j$.

The matrix J is the bordered Hessian matrix for the classical programming problem and must be negative definite if the first-order conditions 2.36 through 2.40 are to be sufficient conditions for a maximum. Therefore, J must be nonsingular and the conditions can be solved for the optimal values of the variables.

By doing so, output supply functions, relating the optimal level of each product to the prices, the expected per-unit patronage refunds, and the quantities of the public goods provided by the cooperative, can be

determined for the typical member patron. Similarly, input demand functions, relating the optimal level of each variable input to the same parameters, can be determined for the member patron. Both sets of these functions can be represented:

$$q_i = q_i\ (P_X,\ P_Y,\ R_c^*,\ Q_G) \qquad i \epsilon X,\ Y \tag{2.42}$$

where P_X is a vector of the prices of the products in set X, P_Y is a vector of the prices of the variable inputs in set Y, and R_c^* is a vector of the expected per-unit patronage refunds on the products in set C.

By horizontally summing the individual output supply functions for product i in set X across all member patrons, a supply function, relating the level of the output supplied by the member patrons to the parameters, can be determined. In a similar manner, a demand function, relating the level of the i-th variable factor in set Y demanded by the member patrons to the parameters, can be determined by horizontally summing the individual input functions for the factor across all member patrons.

Because the per-unit patronage refunds expected by the member patrons may vary from patron to patron, both sets of these functions are best written in terms of a vector of past actual per-unit patronage refunds instead of the expected per-unit patronage refund. These functions can be represented:

$$q_{ic} = q_{ic}\ (P_X,\ P_Y,\ R_c^p,\ Q_G) \qquad i \epsilon X,\ Y \tag{2.43}$$

where R_c^p is a vector of past actual per-unit patronage refunds on the products in set C. Although the cooperative cannot affect current expectations with changes in the current-year per-unit patronage refunds,

it can affect future expectations with them and, thus, can affect future member behavior with them.

Using the Kuhn-Tucker conditions corresponding to the typical nonmember patron, output supply functions and input demand functions for the nonmember patron can be derived. These are similar to those for the typical member patron and represented by 2.42 except that they do not include the expected per-unit patronage refund argument. Horizontally summing these functions across all nonmember patrons, supply and demand functions, relating the levels of nonmember supply and demand to the prices and the quantities of the public goods provided by the cooperative, can be determined. Both sets of these functions can be represented:

$$q_{io} = q_{io}\,(P_X,\ P_Y,\ Q_G) \qquad i \epsilon X,\ Y. \tag{2.44}$$

As prices change, as expected per-unit patronage refunds change, and as the quantities of the public goods provided by the cooperative change, the typical member or nonmember patron will alter his input and output levels to satisfy his first-order conditions. The partial derivative of q_i with respect to any argument in 2.42 shows the effect on the ith output or input of the typical member patron of a one-unit change in that argument.

Evaluation of these partial derivatives begins with differentiating conditions 2.34 through 2.40 totally with respect to all variables and the parameters in 2.42. The corresponding total differentials are:

$$\Psi_1\phi_{i1}dq_1 + \Psi_1\phi_{i2}dq_2 + \ldots + \Psi_1\phi_{ij}dq_j + \ldots + \phi_i d\Psi_1 + dp_i + \left[s + \frac{(1-s)}{(1+d)^\tau}\right] dr_i^* = 0 \quad \text{for all } i\epsilon X_c \tag{2.45}$$

$$\Psi_1\phi_{i1}dq_1 + \Psi_1\phi_{i2}dq_2 + \ldots + \Psi_1\phi_{ij}dq_j + \ldots + \phi_i d\Psi_1 + dp_i = 0 \quad \text{for all } i\epsilon X_o \tag{2.46}$$

$$\Psi_1\phi_{i1}dq_1 + \Psi_1\phi_{i2}dq_2 + \ldots + \Psi_1\phi_{ij}dq_j + \ldots + \phi_i d\Psi_1 - dp_i + \left[s + \frac{(1-s)}{(1+d)^\tau}\right] dr_i^* = 0 \quad \text{for all } i\epsilon X_c \tag{2.47}$$

$$\Psi_1\phi_{i1}dq_1 + \Psi_1\phi_{i2}dq_2 + \ldots + \Psi_1\phi_{ij}dq_j + \ldots + \phi_i d\Psi_1 - dp_i = 0 \quad \text{for all } i\epsilon Y_o \tag{2.48}$$

$$\Psi_1\phi_{i1}dq_1 + \Psi_1\phi_{i2}dq_2 + \ldots + \Psi_1\phi_{ij}dq_j + \ldots + \phi_i d\Psi_1 - d\Psi_{2i} = 0 \quad \text{for all } i\epsilon W_f \tag{2.49}$$

$$\phi_1 dq_1 + \phi_2 dq_2 + \ldots + \phi_j dq_j + \ldots = 0 \tag{2.50}$$

$$-dq_i = 0 \quad \text{for all } i\epsilon W_f. \tag{2.51}$$

In order to solve this system of equations for the unknowns dq_i ($i\epsilon X, Y$), the changes in the parameters in 2.42 are treated as constants. This allows the system to be expressed as the matrix equation 2.52 where:

$$dp_i^* = dp_i + \left[s + \frac{(1-s)}{(1+d)^\tau}\right] dr_i^* \quad \text{for all } i\epsilon X_c,$$

$$= dp_i \quad \text{for all } i\epsilon X_o,$$

$$= -dp_i + \left[s + \frac{(1-s)}{(1+d)^\tau}\right] dr_i^* \quad \text{for all } i\epsilon Y_c,$$

$$= -dp_i \quad \text{for all } i\epsilon Y_o,$$

$$
\left[\begin{array}{cccccccccc|c|cc}
\Psi_1\phi_{11} & \Psi_1\phi_{12} & \cdots & \Psi_1\phi_{1j} & \cdots & & & & \phi_1 & 0 & 0 & \cdots \\
\Psi_1\phi_{21} & \Psi_1\phi_{22} & \cdots & \Psi_1\phi_{2j} & \cdots & & & & \phi_2 & 0 & 0 & \cdots \\
\vdots & \vdots & & \vdots & & & & & \vdots & \vdots & \vdots & \\
\Psi_1\phi_{i1} & \Psi_1\phi_{i2} & \cdots & \Psi_1\phi_{ij} & \cdots & & & & \phi_i & -1 & 0 & \cdots \\
 & & & & & & & & & 0 & -1 & \cdots \\
\vdots & \vdots & & \vdots & & & & & \vdots & \vdots & \vdots & \\
\hline
\phi_1 & \phi_2 & \cdots & \phi_j & \cdots & & & & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots
\end{array}\right]
$$

$$
\left[\begin{array}{c|ccccc}
\phi_1 & 0 & 0 & \cdots & 0 \\
\phi_2 & 0 & 0 & \cdots & 0 \\
 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 0 & \cdots & 0 \\
 & & & & \\
\phi_i & -1 & 0 & \cdots & 0 \\
 & 0 & -1 & \cdots & 0 \\
 & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
 & 0 & 0 & \cdots & -1 \\
\hline
0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0
\end{array}\right]
\left[\begin{array}{c}
dq_1 \\
dq_2 \\
\cdot \\
\cdot \\
\cdot \\
 \\
dq_j \\
\cdot \\
\cdot \\
\cdot \\
 \\
\hline
d\Psi_1 \\
\hline
d\Psi_{21} \\
d\Psi_{22} \\
\cdot \\
\cdot \\
\cdot \\
d\Psi_{2j} \\
\vdots
\end{array}\right]
=
\left[\begin{array}{l}
-dp_1^* - \Psi_1 \sum_{j\in G} \phi_{1j}\, dq_j \\
-dp_2^* - \Psi_1 \sum_{j\in G} \phi_{2j}\, dq_j \\
 \\
 \\
 \\
 \\
-dp_i^* - \Psi_1 \sum_{j\in G} \phi_{ij}\, dq_j \\
 \\
 \\
 \\
 \\
\hline
-\sum_{j\in G} \phi_j\, dq_j \\
\hline
0 \\
0 \\
 \\
 \\
 \\
0
\end{array}\right]
\tag{2.52}
$$

and where, from left to right, the first matrix is the Jacobian matrix J represented by 2.41, the second is the vector of unknowns, and the third is the vector of constants.

Using Cramer's rule, the matrix equation can be solved for the k-th unknown by replacing the k-th column of J by the vector of constants. The solution is then equal to the determinant of this matrix divided by the determinant of J. Letting D represent the determinant of J and letting D_{ij} represent the cofactor of the element in the i-th row and j-th column of J:

$$dq_k = \left[\sum_{i \in X,Y,W_f} (-dp_i^* - \Psi_1 \sum_{j \in G} \phi_{ij}\, dq_j) D_{ik} - \sum_{j \in G} \phi_j\, dq_j\, D_{gk} \right] \Big/ D \quad \text{for all } k \in X,\ Y \tag{2.53}$$

where $-\sum_{j \in G} \phi_j dq_j$ is the g-th element in the vector of constants. The partial derivative $\partial q_k / \partial p_\ell$ is determined by dividing 2.53 by dp_ℓ and setting all other differentials equal to zero. Thus:

$$\frac{\partial q_k}{\partial p_\ell} = - \frac{D_{\ell k}}{D} \quad \text{for all } \ell \in X \tag{2.54}$$

and

$$\frac{\partial q_k}{\partial p_\ell} = \frac{D_{\ell k}}{D} \quad \text{for all } \ell \in Y. \tag{2.55}$$

In general, it is difficult to assess the signs of the partial derivatives of the types 2.54 and 2.55. The cross-effects may be of either sign depending of the particular form of the implicit production function. However, the signs of the own-price effects can be

ascertained. If $k = \ell$, it is always possible to re-order the rows and columns of J so that the k-th row and column are the first row and column. Thus, D_{kk} is a principal minor of J of the order of one less than the order of J. Because it is assumed J is negative definite, D_{kk} and D must be of opposite signs. Thus, 2.54 and 2.55 are respectively positive and negative. In other words, an increase in the price of the k-th output, other prices constant, will have the expected effect of increasing the quantity supplied while an increase of the price of the k-th input, other prices constant, will have the expected effect of decreasing the quantity demanded.

Taking 2.53, dividing by dr^*_ℓ, and setting all other differentials equal to zero:

$$\frac{\partial q_k}{\partial r^*_\ell} = - \left[s + \frac{(1-s)}{(1+d)^T}\right] \frac{D_{\ell k}}{D} \qquad \text{for all } \ell \in C. \tag{2.56}$$

Again, in general, it is difficult to assess the signs of the partial derivatives of the type 2.56. The cross-effects may be of either sign depending upon the particular form of the implicit production function. However, the signs of the own-price effects can be ascertained. D_{kk} is again of the opposite sign of D and the quantity within the brackets is positive. Thus, 2.56 is positive. In other words, an increase in the expected per-unit patronage refund on the k-th output or input, other prices constant, will have the effect of increasing the quantity supplied or demanded.

Although the current behavior of the member patron is not affected by changes in the actual per-unit patronage refunds in the current period,

these changes can affect behavior in future periods. From 2.3, it can be seen that the current per-unit patronage refund on product i affects the expected per-unit patronage refunds on product i in future periods. Thus, for example, the effect on the quantity of i supplied or demanded in the next period of an increase in the current per-unit patronage refund on i is equivalent to:

$$\frac{\partial q_i(t+1)}{\partial r_i} = \frac{\partial q_i(t+1)}{\partial r_i^*(t+1)} \cdot \frac{\partial r_i^*(t+1)}{\partial r_i} \quad . \tag{2.57}$$

If $\partial r_i^*(t+1)/\partial r_i$ is assumed to be positive in sign, so will 2.57.

Again taking 2.53, but dividing by dq_ℓ where $\ell \varepsilon G$, and setting all other differentials equal to zero:

$$\frac{\partial q_k}{\partial q_\ell} = \left[\sum_{i \varepsilon X,Y,W_f} (-\Psi_1 \phi_{i\ell}) D_{ik} - \phi_\ell D_{gk} \right] \Big/ D \qquad \ell \varepsilon G \tag{2.58}$$

As for the cross-effects for 2.54 through 2.56, these effects may be of either sign depending on the particular form of the implicit production function. In other words, nothing can be said, a priori, about the effect of an increase in the ℓ-th public good provided by the cooperative on the quantity of the k-th product supplied or demanded.

This analysis has been for the typical member patron. A similar analysis can be carried out for the typical nonmember patron.

CHAPTER III. COOPERATIVE SUB-MODEL

Activities

The cooperative purchases a set of products X_c from its member and nonmember patrons and sells them a set of variable inputs Y_c which it produces. It also provides member and nonmember patrons with a set of public goods G and produces a set of products Z which it sells to commodity markets. Production of its various outputs necessitates the use of a set of variable inputs V purchased from outside the cooperative association and a set of inputs W_c, the quantities of which are fixed in the short-run.

Objective Function

The cooperative decision-maker is assumed to maximize the total profits of its member patrons:

$$\Pi = \sum_{i \in X} p_i q_{ic} - \sum_{i \in Y} p_i q_{ic} - FCM + DS + PVPR \qquad (3.1)$$

where q_{ic} is the total quantity of product i purchased or sold by the member patrons, FCM is the total fixed costs of the member patrons, DS is the total dividend on member stock, and PVPR is the present value of allocated patronage refunds.

It is assumed that the cooperative decision-maker maximizes the profits of its member patrons for two reasons. First, if the cooperative is viewed as an extension of the member patrons or if it is understood that the purpose of the cooperative is to benefit the member patrons,

maximization of the total profits of the member patrons is consistent with the assumption that member patrons maximize profits. Second, much of the theory of the proprietary firm is based on the assumption of profit maximization. Thus, the assumption that the cooperative maximizes the total profits of the member patrons allows the behavior of the cooperative to be contrasted with that of the proprietary firm.

Production Function

The technology of the cooperative is represented by a production function which, in its implicit form, is written:

$$\Phi (Q_Z, Q_Y, Q_G, Q_X, Q_V, Q_{W_c}) = 0 \qquad (3.2)$$

where Q_Z is a vector of the quantities of each of the products in set Z produced by the cooperative and sold outside the cooperative association, Q_Y is a vector of the quantities of each of the variable factors in set Y produced by the cooperative and sold to patrons, Q_G is a vector of the quantities of each of the public goods in set G produced by the cooperative, Q_X is a vector of the quantities of each of the goods in set X produced by patrons and used in production by the cooperative, Q_V is a vector of the quantities of each of the variable factors in set V used in production by the cooperative and purchased from outside the cooperative association, and Q_{W_c} is a vector of the quantities of each of the fixed factors in set W_c used in production by the cooperative.

The assumptions made concerning this production function are similar to those made concerning the production function of the typical member patron 2.4. It is assumed that the production function 3.2 possesses

continuous first- and second-order partial derivatives which are different from zero for all its nontrivial solutions and that it is written in such a way that the partial derivatives with respect to the outputs are positive and the partial derivatives with respect to the inputs are negative. It is further assumed that 3.2 is subject to diminishing returns such that all one-product production functions obtained from 3.2 by fixing the values of all other outputs are strictly concave.

Distribution of Net Savings

Patronage refunds

In this model, it is assumed that only members receive patronage refunds. A 1966 amendment to the federal income tax law requires that at least 20 percent of allocated patronage refunds must be distributed in cash. In the past, many cooperatives paid in cash only this minimum required by law. More recently, however, pressure from patrons has, in some instances, resulted in increases in the percentage of allocated patronage refunds paid in cash. This pressure stems from the tax considerations of patrons, who are required to pay income taxes not only on that portion of allocated patronage refunds paid in cash but on that portion deferred to the revolving fund.

Iowa law restricts the percentage of allocated patronage refunds paid in cash to not more than 20 percent if there exist unpaid deferred patronage refunds from past years. Typically unpaid deferred patronage refunds from past years exist because the use of revolving funds is a common method of cooperative financing. However, a number of Iowa cooperatives have been able to increase the percentage of allocated

patronage refunds paid in cash by converting deferred patronage refunds from past years into preferred stock, on which a fixed rate of return is paid.

In general, it seems reasonable to assume that the percentage of allocated patronage refunds paid in cash is fixed. Most cooperatives pay in cash the same percentage of allocated patronage refunds year after year. In fact, the percentage of allocated patronage refunds paid in cash may be fixed in the articles of incorporation or the by-laws of the cooperative if not by the laws of the state in which the cooperative is incorporated.

The length of the revolving fund is variable. Most cooperatives, however, feel obligated, once a revolving fund has been set up, to make a concerted attempt at retiring the deferred refunds after a given length of time although this may be difficult or impossible to do. At least, the cooperative can be presumed to have an expectation for the length of the revolving fund. Thus, to assume that the length of the revolving fund in this model is known and fixed should not affect the analysis.

Dividends on stock

The stock of a cooperative is generally divided into common and preferred stock. Usually, purchase of a share of common stock is a condition of membership and members are limited to one share each. Although common stock is voting stock, many cooperatives choose not to pay dividends on shares of common stock so that members receive all returns on a patronage basis.

Usually, the sale of preferred stock is not restricted to members and there is no limit on the number of shares an individual may hold. Preferred stock is nonvoting stock, and, whereas common stock is ownership stock, preferred stock is investment stock. Preferred stock represents money invested into the cooperative for the dividend which it earns.

The Capper-Volstead Act restricts the rate of return on all cooperative stock to no more than 8 percent per annum. State laws may require that the rate of return on cooperative stock be limited to a rate less than 8 percent per annum. In addition, state law may require that the rate of return be fixed so that cooperative stock is, in effect, interest-bearing. Iowa law, for example, requires that the rate of return on preferred stock must be fixed by the articles of incorporation at a rate not exceeding 8 percent per annum.

The rate of return on capital stock in the cooperative in this model is assumed to be fixed. It is also assumed that, in the short-run, the number of shares of stock is fixed. Thus, dividends on stock can be treated as a constant.

Retained savings

The provisions for retained savings vary from state to state. Some states have no provisions for retained savings while others require cooperatives to add savings to their surplus accounts. Iowa law, for example, requires that at least 10 percent of net savings after income taxes must be added to surplus if the surplus account is equal to less than 30 percent of other member equity and that no additions can be made

if the surplus account is equal to more than 50 percent of other member equity.

The typical case appears to be that of the cooperative which has a surplus account which is equal to less than 50 percent of other member equity. Most cooperatives have surplus accounts which are equal to less than 30 percent of other member equity. Of these, most add more than 10 percent of their net savings after income taxes to the surplus account. Of those which have surplus accounts which are equal to more than 30 percent of other member equity, most make additions to their surplus accounts. Many of them add more than 10 percent of their net savings after income taxes to the surplus account.

Coffman [14, p. 31] reasons that since all net savings not allocated as patronage refunds are subject to income taxes, the best interests of both the cooperative and its members are served by minimizing retained savings. However, in light of the actual practices of cooperatives, it seems that the amount of net savings added to the surplus account is not determined by a legal restraint, but by an internal demand for capital. Because a discussion of financial decisions is beyond the scope of this study, it is assumed that the amount of net savings added to the surplus account of the cooperative in this model is determined outside of the model and is treated as a constant inside the model.

Educational fund

Some states require cooperatives to place some of their net savings into an educational fund. Iowa law, for example, requires that between one and five percent of net savings after income taxes must be added to

an educational fund. Since there is no economic advantage to the cooperatives of adding to their educational funds, most Iowa cooperatives add only the minimum required by law. Because most states do not require cooperatives to make additions to educational funds, an educational fund is not included in the analysis in this study.

Income taxes

The cooperative receives its earnings from two sources--member business and nonmember business. If the cooperative distributes net savings from nonmember business to its nonmember patrons in the same way it distributes net savings from member business to its member patrons, it may qualify for tax treatment under section 521 of the Internal Revenue Code of 1954. Under section 521, a cooperative must pay tax on retained savings, but is not required to pay tax on patronage refunds, dividends on stock, and certain sources of income.

A cooperative which does not operate under section 521 must pay taxes on all net savings not allocated as patronage refunds, but may restrict patronage refunds to members. Many cooperatives find it impossible to comply with the strict provisions of section 521 or choose to pay taxes on the net savings on nonmember business so that the net savings from nonmember business can be used to add to the surplus account or to increase the amount of patronage refunds allocated to members. The only restriction placed on the use of the net savings from nonmember business by the cooperative is that it cannot be distributed to members through patronage refunds. Because it is the typical case, it is assumed that the cooperative in this study does not operate under section 521. Instead,

it does not pay patronage refunds to nonmember patrons and must pay income taxes on all net savings not allocated as patronage refunds.

It is assumed that the cooperative must pay an income tax composed of a base sum b for that portion of its taxable income equal to an amount TI_o and an additional tax at a constant marginal rate t for that portion of its taxable income in excess of TI_o. The cooperative's taxable income is equal to its net savings less patronage refunds. Thus, its total tax bill is

$$TX = b + t(NS - PR - TI_o) \tag{3.3}$$

where NS is net savings and PR is patronage refunds and where the term within the parentheses is understood to be nonnegative.[1] The marginal tax rate t is an effective rate which is a function of the state and federal rates. Because state taxes are deductible in computing federal taxes and because federal taxes may be deductible in part in computing state taxes, this rate is not simply the sum of the marginal rates for the state and federal taxes.[2]

Requirements

In this model, dividends on stock, retained savings, and income taxes are requirements which must be met from net savings before patronage refunds can be allocated. A cooperative can elect to pay as much of these requirements as possible out of the net savings from nonmember business although it cannot distribute its net savings from nonmember

[1]The total tax bill 3.3 can alternatively be stated:

$$TX = t_1(NS-PR-a_1) + t_2(NS-PR-a_2) + t_3(NS-PR-a_3) + \ldots \tag{3.3a}$$

where a_i is the value at which the increment t_i is added to the marginal tax rate. As in 3.3, the terms within the parentheses are understood to be nonnegative.

[2]See Proof 5 in Appendix B for the derivation of the marginal rate t for a case in which state and federal taxes are fully deductible from each other.

business to its members through patronage refunds. In this way, the amount allocated as patronage refunds to the member patrons is to the greatest extent possible the net savings on the transactions between the cooperative and its members.

Under this arrangement, the amount allocated as patronage refunds can be expressed:

$$PR = NS_c - R \tag{3.4}$$

where NS_c is the net savings from member business and R is:

$$R = Req - NS_o \geq 0. \tag{3.5}$$

Req is the sum of dividends on stock, retained savings, and income taxes:

$$Req = DS + RS + TX, \tag{3.6}$$

and NS_o is the net savings from nonmember business. Substituting 3.5 into 3.4, the amount allocated as patronage refunds can be expressed:

$$PR = NS - Req. \tag{3.7}$$

Using 3.3, 3.4, and 3.5, 3.6 can be expressed:

$$Req = \frac{DS + RS + b\text{-}t \cdot TI_o}{(1 - t)}. \tag{3.8}$$

Because all of the terms on the right-hand side of 3.8 are constants, Req is a constant.

If the value of 3.8 is greater than the value of the savings from nonmember business, $R > 0$ and deductions to meet the requirements must be taken out of the net savings from member business of the departments in proportion to the patronage refunds (or net savings from member business) of the departments. Because the cooperative cannot distribute its net savings from nonmember business to its members through patronage refunds, it is assumed that Req is sufficiently large to ensure that $R \geq 0$.

Determination of Net Savings and Patronage Refunds

Generally, for the purposes of accounting and management, the enterprises of a multi-product cooperative are divided into departments. For example, a cooperative might consist of a grain department; a feed department; a chemicals, fertilizer, and seed department; a petroleum products department; a building materials and ready mix department; and a merchandise department.

The cooperative in this model is assumed to be organized into departments which purchase products in set X from patrons and use them in the production of products in set Z (marketing departments) and departments which produce products in set Y and sell them to patrons (supply departments). In this way, member patrons who sell products in set X to a marketing department receive patronage refunds on the net savings from the products in set Z which the department sells, and member patrons who purchase products in set Y from a supply department receive patronage refunds on the net savings from the products in set Y which the department sells.

The net savings of each department is determined by subtracting the total cost of the department from the total revenue of the department. The total cost of operating a department may include payments to other departments for products purchased from them and used in the production of the products marketed by the department. Similarly, the total revenue of a department may include receipts from other departments for sales to them. The exchange of products between departments is assumed to conform to the relationships indicated in Figure 3.1.

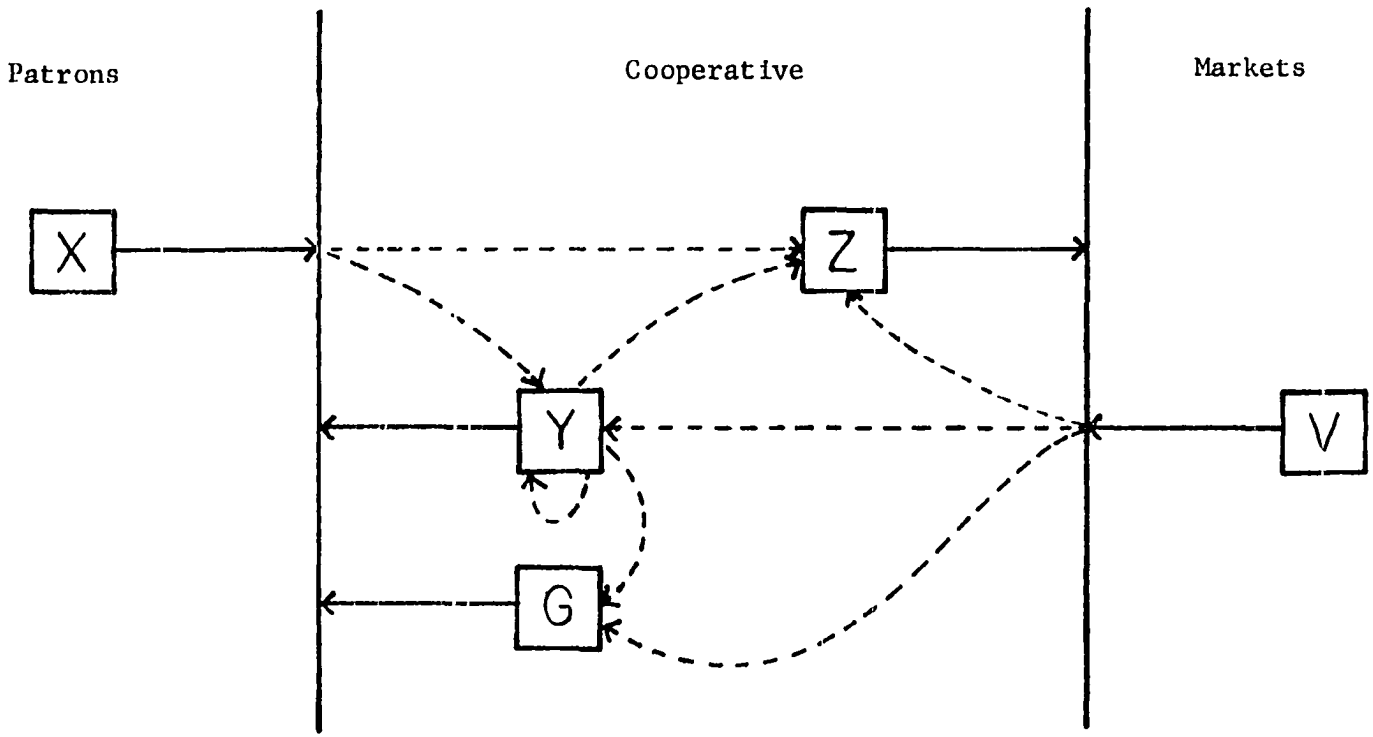

Figure 3.1. Flows of products within the cooperative.

Products in set X, which are purchased from the patrons, are used to produce products in sets Y and Z. Products in set Z are sold to markets outside the cooperative association. Products in set Y are sold to the patrons and are used in the production of products in sets X, Z, and G. In addition, some products in set Y are used in the production of other products which are in set Y but in other departments. Finally, products in set V, which are purchased from markets outside the cooperative, and set W_c are used in the production of products in sets Z, Y, and G.

Thus, the net savings of the k-th marketing department can be expressed:

$$NS_k = \sum_{j\epsilon Z_k} p_j q_j - \sum_{i\epsilon X_k} p_i (q_i - \sum_{j\epsilon Y_c} q_{ij}) - \sum_{i\epsilon Y} \sum_{i\epsilon Z_k} p_i q_{ij}$$

$$- \sum_{i\epsilon V} \sum_{j\epsilon Z_k} p_i q_{ij} - \sum_{i\epsilon W_c} \sum_{j\epsilon Z_k} p_i q_{ij} - C_k \qquad (3.9a)$$

where Z_k is the subset of products in set Z produced in the k-th department, where X_k is the subset of products in set X purchased by the k-th department, where p_i and q_i are the price and quantity of the i-th product, where q_{ij} is the quantity of the i-th product or factor used in the production of the j-th product, and where C_k is the amount of indirect cost allocated to the k-th department. The symbol p_i, for $i\epsilon W_c$, represents the price charged each department for the use of the i-th fixed factor.

The net savings of the k-th supply department can similarly be expressed:

$$NS_k = \sum_{j \epsilon Y_k} p_j q_j - \sum_{i \epsilon X} \sum_{j \epsilon Y_k} p_i q_{ij} - \sum_{\substack{i \epsilon Y \\ i \notin Y_k}} \sum_{j \epsilon Y_k} p_i q_{ij}$$

$$- \sum_{i \epsilon V} \sum_{j \epsilon Y_k} p_i q_{ij} - \sum_{i \epsilon W_c} \sum_{j \epsilon Y_k} p_i q_{ij} - C_k \qquad (3.9b)$$

where Y_k is the subset of products in set Y produced in the k-th department.

Implicit in 3.9 is the assumption that trading between departments is done at market prices. For example, it is assumed that the petroleum products department charges the grain department the same price it charges its patrons for the gasoline the grain department uses in its operations. This seems to be the practice of most cooperatives.

The net savings from member business of the k-th department is defined as the net savings of the k-th department multiplied by the proportion of the total business of the department done with member patrons. Similarly, the net savings from nonmember business of the k-th department is defined as the net savings of the k-th department multiplied by the proportion of the total business of the department done with nonmember patrons.

The total net savings of the cooperative is determined by summing the net savings of the departments over all departments. It can be expressed:

$$NS = \sum_{i \epsilon Z} p_i q_i + \sum_{i \epsilon Y_c} p_i q_i - \sum_{i \epsilon X_c} p_i q_i - \sum_{i \epsilon V} p_i q_i - FCC \qquad (3.10)$$

where FCC represents the total fixed costs of the cooperative. The total net savings from member business, represented by NS_c, is defined as NS multiplied by the proportion of total business done with member patrons.

Similarly, the net savings from nonmember business, represented by NS_o, is defined as NS multiplied by the proportion of total business done with nonmember patrons.

Most cooperatives determine patronage refunds separately for each department. They reason that since operating costs and net savings vary from department to department and since individual members do not make equal use of all departments, departmental determination of patronage refunds is necessary to be fair to all members.

In the method of determining per-unit patronage refunds used by most cooperatives, the per-unit patronage refund for the i-th product in department k is:

$$r_i = \rho_k p_i \tag{3.11}$$

where p_i is the market price of the i-th product and:

$$\rho_k = \frac{NS_{kc} - R_k}{\sum_{j \in D_k} p_j q_{jc}} \tag{3.12}$$

where NS_{kc} is the net savings from member business of department k, R_k is the amount deducted from the net savings from member business of department k to meet the requirements in 3.8, and q_{jc} is the quantity of the j-th product in department k purchased or sold by the member patrons. D_k is the subset X_k if k is a marketing department and Y_k if k is a supply department.

The cooperative or department which determines patronage refunds separately for each product can be treated as a set of departments, each of which consists of only one product. In that case, the method of

determining per-unit patronage refunds represented by 3.11 and 3.12 is equivalent to:

$$r_i = p_i - v_i - f_i \tag{3.13}$$

where v_i is the average variable cost of producing product i and f_i is the average fixed cost allocated to product i. At the other extreme, the case of the cooperative which determines one per-unit patronage refund for all products can be treated as a single department which includes all products.

Some cooperatives may deviate from the method of determining patronage refunds represented by 3.11 and 3.12 when determining patronage refunds for a department which consists of products which are sold to patrons and which rely on products purchased from patrons as major inputs. An example of this type of product is mixed feed (set Y) which is produced by a cooperative by mixing finished grain (set Z) with protein (set V) and sold to patrons.

If the cooperative utilized the method of determining patronage refunds represented by 3.11 and 3.12, member patrons who sold the grain (set X) used in the production of the mixed feed (set Y) would receive the net savings from the grain and the member patrons who purchased the mixed feed would receive the net savings from the mix. However, the member patrons who purchased the mixed feed would not participate at all in the net savings of the grain department since the feed department is assumed to purchase the finished grain at market prices.

Because grain is a major input in the production of mixed feed, some cooperatives producing mixed feed allow member patrons who purchase

the mix to participate in the net savings of the grain department. This is done by giving them shares in the net savings of the grain department based on their purchases of mixed feed. These shares may be partial shares or full shares. Analysis of this type of situation is not included in this study.

Indirect Costs

In this study, it is necessary to distinguish between variable and fixed costs. A variable cost is defined as the cost of a variable input, an input the quantity of which is variable in the short-run. A fixed cost is defined as the cost of a fixed input, an input the available quantity of which is not variable in the short-run.

It is also necessary to distinguish between direct and indirect costs. A direct cost is defined as the cost of an input which is easily traceable to the production or marketing of a product or a segment of business. An indirect cost is defined as the cost of an input which is difficult to trace to a single product or segment because it is common to more than one.

It is important to recognize that variable costs are not synonymous with direct costs and that fixed costs are not synonymous with indirect costs. Some variable costs, such as institutional advertising expenses, are indirect costs in that they cannot be traced to a specific product or segment.[1] On the other hand, some fixed costs, such as the

[1]Institutional advertising is an example of a discretionary fixed cost. Holdren [30, p. 33] introduces the concept of discretionary fixed costs which he defines as "costs which are fixed with respect to output variation, but are decision variables within the functional time period known as the short run."

depreciation on a machine used in producing a single product, are direct costs in that they can be traced to a specific production segment.

The salary of the general manager of a cooperative is a good example of an indirect or joint cost. Whereas the salaries of the assistant managers can easily be traced to the departments which they manage, the salary of the general manager cannot.

Any multi-product cooperative can provide examples of indirect costs. The financial statements of one cooperative, for example, include meetings and travel, loss on disposal of fixed assets, telephone, utilities, insurance, bad debts, interest, advertising, and miscellaneous expenses as indirect costs. In addition, they include some salaries and NYTCO (bonding) expenses, payroll taxes, retirement and insurance, depreciation, property taxes, pest control,[1] truck expense, data processing,[1] general and administrative,[1] OSHA expense,[1] audit and legal,[1] directors' fees,[1] organization costs, dues and subscriptions,[1] and donations[1] as allocated to the administrative department. These expenses are indirect costs in that the costs of operating the administrative department cannot be easily traced to each of the other departments.

The cooperative must assign its indirect costs to its departments so that the net savings of the departments can be determined. The assignment of the indirect costs is important in that the patronage refunds of each department are determined by the net savings of the department.

Indirect costs can be assigned to departments by several methods. First, the cooperative can arbitrarily assign indirect costs to the

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[1]Expenses allocated exclusively to the administrative department.

departments before making its decisions on prices and outputs. These prior cost allocations are pre-determined parameters which may affect decisions on prices and outputs but which are not affected by them.

Second, the cooperative can assign indirect costs to the departments through a basis approach. In a basis approach, an attempt is made to allocate indirect costs among the departments according to the benefits which they receive from the common cost factors. Because there is no way to directly determine how much a particular department benefits from the common cost factors, indirect costs are allocated among the departments by relating them to some other cost factor or basis which can be directly identified with units of output. A cooperative may, for example, allocate indirect costs among its departments in proportion to sales or in proportion to particular direct costs.

If it is to be assumed that the basis selected measures the benefits received from the common cost factors, it should, to the greatest extent possible, be related to their services. However, it should be recognized that any basis by which indirect costs are allocated is necessarily an arbitrary one even if it is a reasonable one.

Ladd [37] offers an instrument approach as a third alternative for allocating indirect costs to departments. In the instrument approach, decisions on the allocation of indirect costs are made simultaneously with decisions on prices and outputs, and the cost allocations, as well as the prices or levels of outputs, serve as instruments for achieving the cooperative's objectives.

It is assumed that there are three types of costs in this model. First, there are direct costs which are assumed to be assigned to the departments corresponding to the products to which they are traceable. These costs are represented in 3.9 as the terms with double summation signs and include direct fixed costs.

Second, there are direct departmental costs, costs which are defined here to be costs which cannot be easily traced to specific products but which can be easily traced to specific departments. As with direct costs, these costs are assumed to be assigned to the departments to which they can be traced. This is in accordance to the principle of service at cost.

Finally, there are the indirect departmental costs, costs which are defined to be costs which cannot be easily traced to specific departments. Because any assignment of these costs would be arbitrary, use of the instrument method in allocating them does not violate the principle of service at cost.

Thus, the amount of indirect costs allocated to the k-th department can be expressed:

$$C_k = C_{Dk} + C_{Ik} \tag{3.14}$$

where C_{Dk} represents the amount of direct departmental costs allocated to the k-th department and where C_{Ik} represents the amount of indirect departmental costs allocated to it. The total amount of indirect departmental costs allocated must equal the total indirect departmental cost C_I, which includes the cost of providing the public goods:

$$C_I = \sum_k C_{Ik}. \tag{3.15}$$

For the purpose of cost allocation, C_I is a constant.

Constraints

The objective function 3.1 which the cooperative is assumed to maximize is subject to several constraints. First, there is the production function, represented by 3.2. Second, there is a set of constraints which ensure use of each fixed factor does not exceed the stock of the factor in possession of the cooperative. If the quantity of the i-th fixed factor used in the production of the j-th product by the cooperative is represented by q_{ij} and the stock of the factor is represented by q_i, the i-th such constraint can be expressed:

$$\sum_{j\epsilon G, Y_c, Z} q_{ij} \leq q_{i0}. \tag{3.16}$$

Finally, there is a constraint which places a limit on the proportion of the cooperative's business which is done with nonmembers. The Capper-Volstead Act stipulates that a cooperative must not deal with nonmembers to an extent exceeding one-half of the value of business done. This constaint can be expressed:

$$\sigma_o \leq \sigma \tag{3.17}$$

where σ represents the maximum proportion of nonmember business allowed by law and where:

$$\sigma_o = \frac{\sum_{i\epsilon C} p_i q_{io}}{\sum_{i\epsilon C} p_i q_i} \tag{3.18}$$

where q_{io} is the quantity of the i-th product bought or sold by non-members and q_i is the total quantity of the i-th product bought or sold.

Lagrangian Function

The objective of the cooperative is to maximize the total profits of its member patrons subject to constraints 3.2, 3.16, and 3.17. Maximization of the total profits of the member patrons (TMP) is equivalent to maximizing the sum of the total private profits of the member patrons (TPP) and the total collective profits of the member patrons (TCP). The total private profits of the member patrons are defined as the difference between the total private revenues of the member patrons (TPR), or the sum of the total revenues of the member patrons exclusive of patronage refunds, and the total private costs of the member patrons (TPC), or the sum of the total costs of the member patrons.

The total collective profits of the member patrons are defined as the difference between the total collective revenues of the member patrons (TCR), or the total revenue of the cooperative multiplied by $s + (1-s)/(1+d_c)^T$, and the total collective costs of the member patrons (TCC), or the total cost of the cooperative multiplied by $s + (1-s)/(1+d_c)^T$. The symbol d_c represents the cooperative's discount rate. The cooperative decision-maker may attempt to set the value of this equal to d, the discount rate of the typical member patron, or he may determine a subjective value which takes into consideration the use of deferred patronage refunds in the cooperative. The total revenue and total cost of the cooperative are multiplied by $s + (1-s)/(1+d_c)^T$ because they affect

the total profits of the member patrons through allocated patronage refunds.

The total collective profits of the cooperative are equal to the total net savings of the cooperative multiplied by $s + (1-s)/(1+d_c)^{\tau}$. Thus, the Lagrangian function corresponding to the problem of the cooperative can be expressed:

$$L = \sum_{i\epsilon X} p_i q_{ic} - \sum_{i\epsilon Y} p_i q_{ic} + \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]NS$$

$$+ \lambda_1 \cdot \Phi (Q_Z, Q_Y, Q_G, Q_X, Q_V, Q_{W_c})$$

$$+ \sum_{i\epsilon W_c} \lambda_{2i}(q_{i0} - \sum_{j\epsilon G, Y_c, Z} q_{ij}) + \lambda_3[\sigma - \sigma_o] \qquad (3.19)$$

where λ_1 is the Lagrange multiplier corresponding to the production function 3.2, the λ_{2i} are the Lagrange multipliers corresponding to the fixed-factor constraints 3.16, and λ_3 is the Lagrange multiplier corresponding to the nonmember-business constraint 3.17.

Kuhn-Tucker Conditions

Among the instruments available to the cooperative are the prices it sets for the products in set C and the quantities of the products in sets G and Z which it produces. The quantities of each of the variable inputs in set V and of each of the fixed factors in set W_c which the cooperative uses in the production of each of the products in sets G, Y_c, and Z are also instruments available to it. In general, it is assumed that the price of a product in set Z may vary inversely with the quantity of the product sold by the cooperative and that the price of a

product in set V may vary directly with the quantity of the input purchased by the cooperative.

The decisions of the cooperative are assumed to be made in two stages. In the first stage, the prices of the products in set C, the quantities of the products in sets G and Z, and the quantities of the inputs in sets V and W_c used in the production of each of the products in sets G, Y_c, and Z are determined. These values determine the volumes of business and the total net savings of the cooperative.

In the second stage, the addition to surplus, the indirect cost allocations, and the patronage refunds are determined. Although it is assumed that the cooperative in this model has perfect knowledge of the marginal cost, supply, and demand functions for the current period, it is assumed that it does not know what its total revenues and total costs are until the end of the accounting period. Thus, it must wait until the end of the accounting period to determine its net savings, its addition to surplus, and its patronage refunds. The values of the patronage refunds for the various products are contingent upon the indirect cost allocations which are also made in this second stage.

Corresponding to the instruments and the Lagrangian function 3.19 is a set of Kuhn-Tucker conditions. These are necessary conditions for a global maximum. They are sufficient conditions for a global maximum if the objective function is concave, the constraints are concave, and the set of feasible solutions is bounded and nonempty.

The Kuhn-Tucker conditions for the problem represented by 3.19 are as follows:

for all $j \epsilon X_c$:

$$\frac{\partial L}{\partial p_j} = q_{jc} + \sum_{i \epsilon X} p_i \frac{\partial q_{ic}}{\partial p_j} - \sum_{i \epsilon Y} p_i \frac{\partial q_{ic}}{\partial p_j} + \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]$$

$$\left[- q_j - \sum_{i \epsilon X_c} p_i \frac{\partial q_i}{\partial p_j} + \sum_{i \epsilon Y_c} p_i \frac{\partial q_i}{\partial p_j}\right] + \sum_{i \epsilon C} \lambda_1 \frac{\partial \Phi}{\partial q_i} \frac{\partial q_i}{\partial p_j}$$

$$- \lambda_3 \frac{\partial \sigma_o}{\partial p_j} \leq 0 \qquad (3.20a)$$

$$\frac{\partial L}{\partial p_j} \cdot p_j = 0 \qquad (3.20b)$$

$$p_j \geq 0 \qquad (3.20c)$$

for all $j \epsilon Y_c$:

$$\frac{\partial L}{\partial p_j} = - q_{jc} + \sum_{i \epsilon X} p_i \frac{\partial q_{ic}}{\partial p_j} - \sum_{i \epsilon Y} p_i \frac{\partial q_{ic}}{\partial p_j} + \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]$$

$$\left[q_j - \sum_{i \epsilon X_c} p_i \frac{\partial q_i}{\partial p_j} + \sum_{i \epsilon Y_c} p_i \frac{\partial q_i}{\partial p_j}\right] + \sum_{i \epsilon C} \lambda_1 \frac{\partial \Phi}{\partial q_i} \frac{\partial q_i}{\partial p_j}$$

$$- \lambda_3 \frac{\partial \sigma_o}{\partial p_j} \leq 0 \qquad (3.21a)$$

$$\frac{\partial L}{\partial p_j} \cdot p_j = 0 \qquad (3.21b)$$

$$p_j \geq 0 \qquad (3.21c)$$

for all $j \epsilon G$:

$$\frac{\partial L}{\partial q_j} = \sum_{i \epsilon X} p_i \frac{\partial q_{ic}}{\partial q_j} - \sum_{i \epsilon Y} p_i \frac{\partial q_{ic}}{\partial q_j} - \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]$$

$$\left[\sum_{i\epsilon X_c} p_i \frac{\partial q_i}{\partial q_j} - \sum_{i\epsilon Y_c} p_i \frac{\partial q_i}{\partial q_j}\right] + \lambda_1 \left[\frac{\partial \Phi}{\partial q_j} + \sum_{i\epsilon C} \frac{\partial \Phi}{\partial q_i} \frac{\partial q_i}{\partial q_j}\right] - \lambda_3 \frac{\partial \sigma_o}{\partial q_j} \leq 0 \qquad (3.22a)$$

$$\frac{\partial L}{\partial q_j} \cdot q_j = 0 \qquad (3.22b)$$

$$q_j \geq 0 \qquad (3.22c)$$

for all $j\epsilon Z$:

$$\frac{\partial L}{\partial q_j} = \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]\left[p_j + q_j \frac{\partial p_j}{\partial q_j}\right] + \lambda_1 \frac{\partial \Phi}{\partial q_j} \leq 0 \qquad (3.23a)$$

$$\frac{\partial L}{\partial q_j} \cdot q_j = 0 \qquad (3.23b)$$

$$q_j \geq 0 \qquad (3.23c)$$

for all $i\epsilon V$; $j\epsilon G, Y_c, Z$:

$$\frac{\partial L}{\partial q_{ij}} = \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]\left[-p_i - q_i \frac{\partial p_i}{\partial q_i}\right] + \lambda_1 \frac{\partial \Phi}{\partial q_{ij}} \leq 0 \qquad (3.24a)$$

$$\frac{\partial L}{\partial q_{ij}} \cdot q_{ij} = 0 \qquad (3.24b)$$

$$q_{ij} \geq 0 \qquad (3.24c)$$

for all $i\epsilon W_c$; $j\epsilon G, Y_c, Z$:

$$\frac{\partial L}{\partial q_{ij}} = \lambda_1 \frac{\partial \Phi}{\partial q_{ij}} - \lambda_{2i} \leq 0 \qquad (3.25a)$$

$$\frac{\partial L}{\partial q_{ij}} \cdot q_{ij} = 0 \qquad (3.25b)$$

$$q_{ij} \geq 0 \qquad (3.25c)$$

for λ_1:

$$\frac{\partial L}{\partial \lambda_1} = \Phi(Q_Z, Q_Y, Q_G, Q_X, Q_V, Q_{W_c}) = 0 \tag{3.26}$$

for λ_{2i}, $i \in W_c$:

$$\frac{\partial L}{\partial \lambda_{2_i}} = q_{i0} - \sum_{j \in G, Y_c, Z} q_{ij} \geq 0 \tag{3.27a}$$

$$\frac{\partial L}{\partial \lambda_{2i}} \cdot \lambda_{2i} = 0 \tag{3.27b}$$

$$\lambda_{2i} \geq 0 \tag{3.27c}$$

for λ_3:

$$\frac{\partial L}{\partial \lambda_3} = \sigma - \sigma_o \geq 0 \tag{3.28a}$$

$$\frac{\partial L}{\partial \lambda_3} \cdot \lambda_3 = 0 \tag{3.28b}$$

$$\lambda_3 \geq 0. \tag{3.28c}$$

No attempt is made in this chapter to interpret these conditions. That task is reserved for the next chapter.

CHAPTER IV. ANALYSIS

Interpretation of Lagrange Multipliers

Before the Kuhn-Tucker conditions can be interpreted, it is necessary to interpret the Lagrange multipliers. It should be repeated that, in general, the value of a Lagrange multiplier at a solution indicates how much the value of the objective function will change given a one-unit change in the corresponding constraint constant. Thus, λ_{2i} can be interpreted as the imputed value or shadow price of the i-th fixed factor:

$$\lambda_{2i} = \frac{\partial \Pi}{\partial q_{i0}} \geq 0 \qquad i \varepsilon W_c. \tag{4.1}$$

It indicates how much the profits of the member patrons would increase with an increase in the i-th fixed factor available.

Similarly, the value of the Lagrange multiplier λ_3 indicates how the profits of the member patrons would increase with a one-unit change in the maximum proportion of nonmember business allowed by law:

$$\lambda_3 = \frac{\partial \Pi}{\partial \sigma} \geq 0. \tag{4.2}$$

A change in the j-th price or a change in the quantity of the j-th public good provided by the cooperative may affect the proportion of the cooperative's business which is done with nonmembers as indicated by the partial derivative $\partial \sigma_o / \partial p_j$ for $j \varepsilon C$ or the partial derivative $\partial \sigma_o / \partial q_j$ for $j \varepsilon G$. If the nonmember-business constraint 3.20 is binding, a change in the proportion of the cooperative's business done with

nonmembers caused by a change in the j-th public good provided by the cooperative must be offset by another change.

Thus, the product $-\lambda_3(\partial\sigma_o/\partial p_j)$ for $j\varepsilon C$ represents the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by a variation in p_j. Similarly, the product $\lambda_3(\partial\sigma_o/\partial q_j)$ for $j\varepsilon G$ represents the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by a variation in q_j. Depending upon the signs of $\partial\sigma_o/\partial p_j$ for $j\varepsilon C$ and $\partial\sigma_o/\partial q_j$ for $j\varepsilon G$ and whether the value of λ_3 is zero or positive, $\lambda_3(\partial\sigma_o/\partial p_j)$ for $j\varepsilon C$ and $\lambda_3(\partial\sigma_o/\partial q_j)$ for $j\varepsilon G$ may be positive, negative, or zero in value.

The same difficulties which arose in the interpretation of Ψ_1 for the typical member patron arise in the interpretation of λ_1 for the cooperative. Under certain conditions, it is possible to mathematically demonstrate that at a maximum $-\lambda_1(\partial\bar{\Phi}/\partial q_i)$ for an output i is equal to the marginal variation in the profits of the member patrons arising from a change in the quantity of the output produced by the cooperative and that $\lambda_1(\partial\bar{\Phi}/\partial q_i)$ for an input i is equal to the marginal variation in the profits of the member patrons arising from a change in the quantity of the input used in production by the cooperative.[1]

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[1]See Proofs 3 and 4 in Appendix B. Although the appendix only contains proofs for an input in set X_c and an output in set Y_c, a corresponding proof can easily be performed for any other input or output by using similar logic.

There is a considerable difference between the interpretation of $-\lambda_1(\partial\Phi/\partial q_i)$ for an output i as the marginal variation in the profits of the member patrons arising from a change in the quantity of the output produced by the cooperative and the interpretation of $-\Psi_1(\partial\Phi/\partial q_i)$ for an output i as the marginal cost of producing the output in the case of the typical member patron. The marginal variation in the profits of the member patrons arising from a change in the quantity of output i produced by the cooperative includes the effect on the profits of the member patrons of the change in the quantities of the inputs used by the cooperative in producing the output, the effect on the profits of the member patrons of the change in the ratio of nonmember business to total business resulting from the change in the quantities of the inputs used by the cooperative in producing the output, and the marginal opportunity cost of using fixed factors of production.[1]

Similarly, there is a considerable difference between the interpretation of $\lambda_1(\partial\Phi/\partial q_i)$ for an input i as the marginal variation in the profits of the member patrons arising from a change in the quantity of the input used in production by the cooperative and the interpretation of $\Psi_1(\partial\phi/\partial q_i)$ for an input i as the marginal revenue (value) product of the input in the case of the typical member patron. The marginal variation in the profits of the member patrons arising from a change in the quantity of input i used in production by the cooperative includes the effect on the profits of the member patrons of the change in the

[1] These effects are considered in Proof 3 through the inclusion of the terms in A.19.

quantities of the outputs produced by the cooperative and the effect on the profits of the member patrons of the change in the ratio of nonmember business to total business resulting from the change in the quantities of the outputs produced by the cooperative.[1]

General Model

After the interpretation of the Lagrange multipliers, it is possible to begin interpreting the Kuhn-Tucker conditions represented by 3.20 through 3.28. As with the patron sub-models, whenever there is a set of an (a), (b), and (c) condition, the (c) condition requires that the instrument to which the conditions correspond must be nonnegative. In most cases, it will be useful to assume that the value of the instrument is positive. If this is so, the (b) condition guarantees that the (a) condition is satisfied as an equality.

If the cooperative offers a positive price for the j-th product in set X_c, condition 3.20a is satisfied as an equality and can be rewritten:

$$(p_j + q_{jc}\frac{\partial p_j}{\partial q_{jc}})\frac{\partial q_{jc}}{\partial p_j} + \sum_{\substack{i\in X \\ i\neq j}} p_i \frac{\partial q_{ic}}{\partial p_j} - \sum_{i\in Y} p_i \frac{\partial q_{ic}}{\partial p_j} - \left[s + \frac{(1-s)}{(1+d_c)^T}\right]$$

$$\left[(p_j + q_j \frac{\partial p_j}{\partial q_j})\frac{\partial q_j}{\partial p_j} + \sum_{\substack{i\in X_c \\ i\neq j}} p_i \frac{\partial q_i}{\partial p_j} - \sum_{i\in Y_c} p_i \frac{\partial q_i}{\partial p_j}\right] + \sum_{i\in C} \lambda_1 \frac{\partial \Phi}{\partial q_i}\frac{\partial q_i}{\partial p_j}$$

$$- \lambda_3 \frac{\partial \sigma_o}{\partial p_j} = 0 \qquad \text{for all } j\in X_c. \tag{4.3}$$

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[1] These effects are considered in Proof 4 through the inclusion of the terms in A.28.

The partial derivatives $\partial q_{ic}/\partial p_j$ and $\partial q_i/\partial p_j$, where $j \epsilon X_c$, are determined by summing the slopes of the individual supply or demand functions across the member patrons and all patrons, respectively. The slopes of the typical member patron's supply and demand functions with respect to changes in the j-th price, $j \epsilon X_c$, are represented by 2.54, in which k=i and l=j.

The term $p_j + q_{jc}(\partial p_j/\partial q_{jc})$ can be interpreted as the marginal variation in total private revenues from the j-th product. Thus, $[p_j + q_{jc}(\partial p_j/\partial q_{jc})]\, \partial q_{jc}/\partial p_j$ can be interpreted as the marginal variation in total private revenues from the j-th product arising from output shifts which are induced by a variation in the j-th price (dp_j). This effect can be represented by $(\partial TPR/\partial q_{jc})(\partial q_{jc}/\partial p_j)$. The term $\sum_{\substack{i \epsilon X \\ i \neq j}} p_i(\partial q_{ic}/\partial p_j)$ can be interpreted as the marginal variation in total private revenues from all other products in set X arising from output shifts which are induced by dp_j. This effect can be represented by $\sum_{\substack{i \epsilon X \\ i \neq j}} (\partial TPR/\partial q_{ic})(\partial q_{ic}/\partial p_j)$. Similarly, the term $\sum_{i \epsilon Y} p_i(\partial q_{ic}/\partial p_j)$ can be interpreted as the marginal variation in total private costs arising from shifts in factor use which are induced by dp_j. This effect can be represented by $\sum_{i \epsilon Y} (\partial TPC/\partial q_{ic})(\partial q_{ic}/\partial p_j)$.

Letting s' represent $s + (1-s)/(1+d_c)^{\tau}$, the term $s'[p_j + q_j(\partial p_j/\partial q_j)]$ can be interpreted as the marginal variation in total

collective costs from the j-th product.[1] Thus, $s'[p_j + q_j(\partial p_j/\partial q_j)]$ can be interpreted as the marginal variation in total collective costs from the j-th product arising from changes in the quantities supplied which are induced by dp_j. This effect can be represented by $(\partial TCC/\partial q_j)(\partial q_j/\partial p_j)$. The term $s' \sum_{\substack{i\in X_c \\ i\neq j}} p_i(\partial q_i/\partial p_j)$ can be interpreted as the marginal variation in total collective costs from all other products in set X_c arising from changes in the quantities supplied which are induced by dp_j. This effect can be represented by $\sum_{\substack{i\in X_c \\ i\neq j}} (\partial TCC/\partial q_i)(\partial q_i/\partial p_j)$. Similarly, the term $s' \sum_{i\in Y_c} p_i(\partial q_i/\partial p_j)$ can be interpreted as the marginal variation in total collective revenues from the products in set Y_c arising from changes in the quantities demanded which are induced by dp_j. This effect can be represented by $\sum_{i\in Y_c} (\partial TCR/\partial q_i)(\partial q_i/\partial p_j)$.

In the discussion of the Lagrange multipliers, $-\lambda_1(\partial\Phi/\partial q_i)$ for an output i was interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of the output

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[1] The existence of the term $s + (1-s)(1+d_c)^{\tau}$ in the Kuhn-Tucker conditions suggests a possible conflict between short-run and long-run objectives or between member patrons and the cooperative decision-maker. Maximization of the profits of the member patrons including the present value of patronage refunds is not the same as maximization of the profits of the member patrons including the cash value of patronage refunds. A change from the former to the latter might result in more of the profits of the member patrons taking the form of patronage refunds. This would result in more capital for long-run investment. A second source of conflict is the amount of retained savings, RS in 3.6. The differences between short-run and long-run objectives are not examined here. They involve decisions on investment and financing and are beyond the scope of this study.

produced by the cooperative. Thus, the term $-\sum_{i\epsilon Y_c} \lambda_1(\partial\Phi/\partial q_i)(\partial q_i/\partial p_j)$ can be interpreted as the marginal variation in the profits of the member patrons from the production of the products in set Y_c arising from changes in the quantities demanded which are induced by dp_j. This effect can be represented by $\sum_{i\epsilon Y_c} (\partial TMP/\partial q_i)(\partial q_i/\partial p_j)$. Similarly, $\lambda_1(\partial\Phi/\partial q_i)$ for an input i was interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of the input used in production by the cooperative. Thus, $\sum_{i\epsilon X_c} \lambda_1(\partial\Phi/\partial q_i)(\partial q_i/\partial p_j)$ can be interpreted as the marginal variation in the profits of the member patrons from the use in production of the products in set X_c arising from changes in the quantities supplied which are induced by dp_j. This effect can be represented by $\sum_{i\epsilon X_c} (\partial TMP/\partial q_i)(\partial q_i/\partial p_j)$.

The term $\lambda_3(\partial\sigma_o/\partial p_j)$ was interpreted as the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dp_j. This effect can be represented by $(\partial TMP/\partial\sigma_o)(\partial\sigma_o/\partial p_j)$.

Thus, if the cooperative offers a positive price for the j-th product in set X_c, the following equality must be satisfied for a maximum:

$$\left(\sum_{i\epsilon X} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} - \sum_{i\epsilon Y} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j}\right) + \left(\sum_{i\epsilon Y_c} \frac{\partial TCR}{\partial q_i} \frac{\partial q_i}{\partial p_j} - \sum_{i\epsilon X_c} \frac{\partial TCC}{\partial q_i} \frac{\partial q_i}{\partial p_j}\right)$$

$$+ \left(\sum_{i \epsilon X_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial p_j} - \sum_{i \epsilon Y_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial p_j} \right) - \frac{\partial TMP}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial p_j} = 0$$

for all $j \epsilon X_c$. (4.4)

This is equivalent to stating that, for a maximum, the sum of the marginal variation in total private profits arising from input and output shifts induced by dp_j; the marginal variation in total collective profits arising from changes in the quantities supplied and demanded induced by dp_j; the marginal variation in the profits of the member patrons arising from changes in the cooperative's production induced by dp_j; and the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business induced by dp_j must equal zero.

Condition 4.4 is more complicated than the analogous condition for the proprietary firm. In general, the optimality condition corresponding to 4.4 for the proprietary firm contains two sets of revenue and cost terms - one set associated with the sales and purchases (market activities) of the firm, the other associated with the production activities of the firm.[1] Terms analogous to these appear within the second and third sets of parentheses in 4.4.

[1]For example, Holdren [30, p. 127] presents an optimality condition for the retail firm which can be rewritten:

$$\sum_{i=1}^{n} \frac{\partial TR}{\partial q_i} \frac{\partial q_i}{\partial p_j} - \sum_{i=1}^{n} \frac{\partial TC}{\partial q_i} \frac{\partial q_i}{\partial p_j} = 0 \qquad (4.5)$$

where n is the number of products sold and where TR and TC represent the firm's total revenue and total cost, respectively. The revenue terms correspond to the market activities of the firm and the cost terms correspond to the production activities of the firm.

However, because the cooperative attempts to maximize the sum of total private profits and total collective profits of its member patrons, the optimality condition for the cooperative also includes a set of revenue and cost terms corresponding to the market activities of the member patrons. These are represented by the terms within the first set of parentheses in 4.4. In addition, 4.4 includes the term representing the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dp_j.

The purchase by member patrons of products in set X_c from the cooperative affects total collective revenues as well as total private costs. Similarly, the sales by member patrons of products in set Y_c to the cooperative affects total collective costs as well as total private revenues. If the proportion of the patronage refunds allocated which are paid in the current period is equal to one (s=1) or if the cooperative sets its discount rate equal to zero (d_c=0), these effects cancel each other out and 4.4 can be rewritten:

$$\left(\sum_{i\varepsilon X_o} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} - \sum_{i\varepsilon Y_o} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j}\right) + \left(\sum_{i\varepsilon Y_c} \frac{\partial TCR}{\partial q_{io}} \frac{\partial q_{io}}{\partial p_j}\right.$$

$$\left. - \sum_{i\varepsilon X_c} \frac{\partial TCC}{\partial q_{io}} \frac{\partial q_{io}}{\partial p_j}\right) + \left(\sum_{i\varepsilon X_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial p_j} - \sum_{i\varepsilon Y_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial p_j}\right)$$

$$- \frac{\partial TMP}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial p_j} = 0 \qquad \text{for all } j\varepsilon X_c \tag{4.6}$$

where X_o is the subset of products in set X which are produced by the member patrons and sold outside the cooperative association and where Y_o

is the subset of products in set Y which are purchased by the member patrons from outside the cooperative association. The interpretation of 4.6 is identical to that of 4.4 except that the marginal variation in total private profits arising from member patron input and output shifts is limited to that from products purchased from sellers or sold to buyers outside the cooperative association and that the marginal variation in total collective costs arising from changes in the quantities supplied and demanded is limited to that from the quantities supplied or demanded by nonmember patrons.

If in addition to assuming that $s = 1$ or $d_c = 0$, it is assumed that the cooperative does not do business with nonmembers, 4.4 reduces to:

$$\left(\sum_{i \epsilon X_o} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} - \sum_{i \epsilon Y_o} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} \right) + \left(\sum_{i \epsilon X_c} \frac{\partial TMP}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} \right.$$

$$\left. - \sum_{i \epsilon Y_c} \frac{\partial TMP}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial p_j} \right) = 0 \qquad \text{for all } j \epsilon X_c . \tag{4.7}$$

This is equivalent to stating that, for a maximum, the sum of the marginal variation in total private profits arising from input and output shifts (in the quantities of the products the member patrons purchase from sellers or sell to buyers outside the cooperative association) induced by dp_j and the marginal variation in the profits of the member patrons arising from changes in the cooperative's production induced by dp_j must equal zero.

Interpretation of condition 3.21a is very similar to that of condition 3.20a. If the cooperative charges a positive price for the j-th product in set Y_c, condition 3.21a is satisfied as an equality and can

be rewritten:

$$- (p_j + q_{jc} \frac{\partial p_j}{\partial q_{jc}}) \frac{\partial q_{jc}}{\partial p_j} + \sum_{i \epsilon X} p_i \frac{\partial q_{ic}}{\partial p_j} - \sum_{\substack{i \epsilon Y \\ i \neq j}} p_i \frac{\partial q_{ic}}{\partial p_j}$$

$$+ [s + \frac{(1-s)}{(1+d_c)^{\tau}}][(p_j + q_{jc} \frac{\partial p_j}{\partial q_j}) \frac{\partial q_j}{\partial p_j} - \sum_{i \epsilon X_c} p_i \frac{\partial q_i}{\partial p_j}$$

$$+ \sum_{\substack{i \epsilon Y_c \\ i \neq j}} p_i \frac{\partial q_i}{\partial p_j}] + \sum_{i \epsilon C} \lambda_1 \frac{\partial \Phi}{\partial q_i} \frac{\partial q_i}{\partial p_j} - \lambda_3 \frac{\partial \sigma_o}{\partial p_j} = 0$$

for all $j \epsilon Y_c$. (4.8)

The partial derivatives $\partial q_{ic}/\partial p_j$ and $\partial q_i/\partial p_j$, where $i \epsilon Y_c$, are determined by summing the slopes of the individual supply or demand functions across the member patrons and all patrons, respectively. The slopes of the typical member patron's supply and demand functions with respect to changes in the j-th price, $j \epsilon Y_c$, are represented by 2.55, in which k = i and l = j.

The term $p_j + q_{jc}(\partial p_j/\partial q_{jc})$ can be interpreted as the marginal variation in total private costs from the j-th product. Thus, $[p_j + q_{jc}(\partial p_j/\partial q_{jc})]\, \partial q_{jc}/\partial p_j$ can be interpreted as the marginal variation in total private costs from the j-th product arising from shifts in factor use which are induced by dp_j. This effect can be represented by $(\partial TPC/\partial q_{jc})(q_{jc}/\partial p_j)$.

The term $s' [p_j + q_j(\partial p_j/\partial q_j)]$ can be interpreted as the marginal variation in total collective revenues from the j-th product. Thus, $s' [p_j + q_j(\partial p_j/\partial q_j)]\, \partial q_j/\partial p_j$ can be interpreted as the marginal variation in total collective revenues from the j-th product arising from

changes in the quantities demanded which are induced by dp_j. This effect can be represented by $(\partial TCR/\partial q_j)(\partial q_j/\partial p_j)$.

All other terms in 4.8 appear in 4.3 and have the same interpretations as they did in 4.3. Thus, if the cooperative charges a positive price for the j-th product in set in Y_c, 4.4 for $j \varepsilon Y_c$ must be satisfied for a maximum. It has the same interpretation as it did for $j \varepsilon X_c$. Expressions analogous to 4.6 and 4.7 can be derived for $j \varepsilon Y_c$. However, because of the degree of similarity between 4.3 and 4.8, they are not presented here.

If the cooperative produces a positive quantity of the j-th public good in set G, condition 3.22a is satisfied as an equality. The partial derivatives $\partial q_{ic}/\partial q_j$ and $\partial q_i/\partial q_j$, where $j \varepsilon G$, are determined by summing the slopes of the individual supply or demand functions across the member patrons and all patrons, respectively. The slopes of the typical member patron's supply and demand functions with respect to changes in the j-th public good are represented by 2.58, in which k = i and l = j.

The term $\sum_{i \varepsilon X} p_i(\partial q_{ic}/\partial q_j)$ can be interpreted as the marginal variation in total private revenues from all products in set X arising from output shifts which are induced by a variation in the level of the j-th public good (dq_j). This effect can be represented by $\sum_{i \varepsilon X} (\partial TPR/\partial q_{ic})(\partial q_{ic}/\partial q_j)$. Similarly, the term $\sum_{i \varepsilon Y} p_i(\partial q_{ic}/\partial q_j)$ can be interpreted as the marginal variation in total private costs from all products in set Y arising from shifts in factor use which are induced by dq_j. This effect can be represented by $\sum_{i \varepsilon Y} (\partial TPC/\partial q_{ic})(\partial q_{ic}/\partial q_j)$.

The term $s' \sum_{i \epsilon X_c} p_i(\partial q_i/\partial q_j)$ can be interpreted as the marginal variation in total collective costs from the products in x_c arising from changes in the quantities supplied which are induced by dq_j. This effect can be represented by $\sum_{i \epsilon X_c} (\partial TCC/\partial q_i)(\partial q_i/\partial q_j)$. Similarly, the term $s' \sum_{i \epsilon Y_c} p_i(\partial q_i/\partial q_j)$ can be interpreted as the marginal variation in total collective revenues from the products in Y_c arising from changes in the quantities demanded which are induced by dq_j. This effect can be represented by $\sum_{i \epsilon Y_c} (\partial TCR/\partial q_i)(\partial q_i/\partial q_j)$.

From the discussion of the Lagrange multipliers, $-\lambda_1(\partial \Phi/\partial q_i)$ for an output i can be interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of the output produced by the cooperative. For i = j, this effect can be represented by $(\partial TMP/\partial q_j)$. The term $- \sum_{i \epsilon Y_c} \lambda_1(\partial \Phi/\partial q_i)(\partial q_i/\partial q_j)$ can be interpreted as the marginal variation in the profits of the member patrons from the production of products in set Y_c arising from changes in the quantities demanded which are induced by dq_j. This effect can be represented by $\sum_{i \epsilon Y_c} (\partial TMP/\partial q_i)(\partial q_i/\partial q_j)$. Similarly, $\lambda_1(\partial \Phi/\partial q_i)$ for an input i can be interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of the input used in production by the cooperative. Thus, $\sum_{i \epsilon X_c} \lambda_1(\partial \Phi/\partial q_i)(\partial q_i/\partial q_j)$ can be interpreted as the marginal variation in the profits of the member patrons from the use in production of products in set X_c arising from

changes in the quantities supplied which are induced by dq_j. This effect can be represented by $\sum_{i \epsilon X_c} (\partial TMP/\partial q_i)(\partial q_i/\partial q_j)$.

The term $\lambda_3(\partial \sigma_o/\sigma q_j)$ was interpreted as the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dq_j. This effect can be represented by $(\partial TMP/\partial \sigma_o)(\partial \sigma_o/\partial q_j)$.

Thus, if the cooperative provides a positive quantity of the j-th public good, the following equality must be satisfied for a maximum:

$$\left(\sum_{i \epsilon X} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} - \sum_{i \epsilon Y} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} \right) + \left(\sum_{i \epsilon Y_c} \frac{\partial TCR}{\partial q_i} \frac{\partial q_i}{\partial q_j} - \right.$$

$$\left. - \sum_{i \epsilon X_c} \frac{\partial TCC}{\partial q_i} \frac{\partial q_i}{\partial q_j} \right) + \frac{\partial TMP}{\partial q_j} + \left(\sum_{i \epsilon X_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial q_j} - \sum_{i \epsilon Y_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_i}{\partial q_j} \right)$$

$$- \frac{\partial TMP}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial q_j} = 0 \quad \text{for all } j \epsilon G. \tag{4.9}$$

This is equivalent to stating that, for a maximum, the sum of the marginal variation in total private profits arising from input and output shifts induced by dq_j; the marginal variation in total collective profits arising from changes in the quantities supplied and demanded induced by q_j; the marginal variation in the profits of the member patrons arising from changes in the cooperative's production induced by dq_j (including dq_j itself); and the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business induced by dq_j must equal zero.

It is apparent from 4.9 that if the j-th public good is a non-excludable public good, its effect on nonmember patrons must be taken into consideration in the determination of the optimal level of the public good. If the public good is an excludable public good and nonmember patrons are excluded from using it, 4.9 can be rewritten:

$$\left(\sum_{i \epsilon X} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} - \sum_{i \epsilon Y} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} \right) + \left(\sum_{i \epsilon Y_c} \frac{\partial TCR}{\partial q_i} \frac{\partial q_{ic}}{\partial q_j} \right.$$

$$\left. - \sum_{i \epsilon X_c} \frac{\partial TCC}{\partial q_i} \frac{\partial q_{ic}}{\partial q_j} \right) + \frac{\partial TMP}{\partial q_j} + \left(\sum_{i \epsilon X_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_{ic}}{\partial q_j} - \sum_{i \epsilon Y_c} \frac{\partial TMP}{\partial q_i} \frac{\partial q_{ic}}{\partial q_j} \right)$$

$$- \frac{\partial TMP}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial q_i} = 0 \quad \text{for all } j \epsilon G. \tag{4.10}$$

The interpretation of 4.10 is identical to that of 4.9 except that the marginal variation in total collective profits arising from changes in the quantities supplied and demanded and the marginal variation in the profits of the member patrons arising from changes in the cooperative's production is limited to that from quantities supplied or demanded by member patrons.

The marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business induced by q_j remains in 4.10 because the level of the j-th public good can affect the ratio through the quantities supplied and demanded by member patrons. If the cooperative does not serve nonmember patrons, this effect would be absent from 4.10.

If it is assumed that the cooperative does not serve nonmembers and that $s = 1$ or $d_c = 0$, 4.9 reduces to:

$$\left(\sum_{i \epsilon X_o} \frac{\partial TPR}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} - \sum_{i \epsilon Y_o} \frac{\partial TPC}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} \right) + \frac{\partial TMP}{\partial q_j} + \left(\sum_{i \epsilon X_c} \frac{\partial TMP}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} \right.$$

$$\left. - \sum_{i \epsilon Y_c} \frac{\partial TMP}{\partial q_{ic}} \frac{\partial q_{ic}}{\partial q_j} \right) = 0 \quad \text{for all } j \epsilon G. \tag{4.11}$$

This is equivalent to stating that, for a maximum, the sum of the marginal variation in total private profits arising from input and output shifts (in the quantities of the products the member patrons purchase from sellers or sell to buyers outside the cooperative association) induced by dq_j and the marginal variation in the profits of the member patrons arising from changes in the cooperative's production induced by dq_j (including dq_j itself) must equal to zero. This is similar to the interpretation given to 4.7 for dp_j, $j \epsilon X_c$.

If the cooperative produces a positive quantity of the j-th product in set Z, condition 3.23a is satisfied as an equality. The term $p_j + q_j (\partial p_j / \partial q_j)$ is the marginal revenue to the cooperative from the j-th product. Again, from the discussion of the Lagrange multipliers, $-\lambda_1 (\partial \Phi / \partial q_j)$ for an output j can be interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of the output produced by the cooperative (dq_j). Thus, 3.23a is equivalent to stating that, for a maximum, the marginal revenue to the cooperative, multiplied by $s + (1-s)/(1+d_c)^{\tau}$, must equal the marginal variation in the profits of the member patrons arising from dq_j.

If the cooperative uses a positive quantity of the i-th variable input in set V in the production of the j-th product in set G, Y_c, or Z, condition 3.24a is satisfied as an equality. The term $p_i + q_i (\partial p_i / \partial q_i)$

is the marginal factor cost to the cooperative of using the i-th variable input. The term $\lambda_1(\partial\Phi/\partial q_{ij})$ can be interpreted as the marginal variation in the profits of the member patrons from a change in the quantity of input i used in the production of output j by the cooperative (dq_{ij}). Thus, 3.24a is equivalent to stating that, for a maximum, the marginal variation in the profits of the member patrons from dq_{ij} must equal the marginal factor cost to the cooperative of using the input, multiplied by $s + (1-s)/(1+d_c)^T$. Obviously, this result implies that the marginal variation in the profits of the member patrons from dq_{ij} must be equal for all j for a maximum.

If the cooperative uses a positive quantity of the i-th fixed input in set W_c in the production of the j-th product in set G, Y_c, or Z, condition 3.25a is satisfied as an equality. Again, the term $\lambda_1(\partial\Phi/\partial q_{ij})$ can be interpreted as the marginal variation in the profits of the member patrons from a change in the quantity of input i used in the production of output j by the cooperative. From the discussion of the Lagrange multipliers, λ_{2i} can be interpreted as the imputed value or shadow price to the member patrons of the i-th fixed factor. Thus, 3.25a is equivalent to stating that, for a maximum, the marginal variation in the profits of the member patrons from dq_{ij} must equal the imputed value to the cooperative of the factor. Again, this result implies that the marginal variation in the profits of the member patrons from dq_{ij} must be equal for all j for a maximum.

Condition 3.26 is simply a restatement of the cooperative's production function. Conditions 3.27a through 3.27c correspond to the

cooperative's fixed-factor constraints 3.16. Their interpretation is similar to that given to the corresponding conditions 2.13a through 2.13c for the typical member patron as represented by 2.21a and 2.21b.

Conditions 3.28a through 3.28c correspond to the nonmember-business constraint 3.17. Condition 3.28a is a restatement of 3.17. Condition 3.28c requires that the value of λ_3 must be nonnegative. If the value of λ_3 is positive, condition 3.28b guarantees that 3.28a is an equality. If 3.28a is satisfied as a strict inequality, i.e., if there is slack in the nonmember business constraint, 3.28b guarantees that λ_3 is equal to zero.

For most agricultural cooperatives, it is expected that the nonmember-business constraint will not be binding. Generally only cooperatives which sell petroleum products in metropolitan areas and cooperatives operating under section 521 of the Internal Revenue Code might be expected to have binding nonmember-business constraints. Thus, in general, λ_3 and the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by a variation in a price or in the quantity of a public good are equal to zero.

It is apparent that the Kuhn-Tucker conditions 3.20 through 3.28 are very complex. In addition, there is a great amount of information which is necessary to evaluate them. Among the values which the cooperative decision-maker must know are:

$$\partial q_{ic}/\partial p_j \qquad \begin{matrix} i \varepsilon X, Y \\ j \varepsilon C \end{matrix}$$

$\partial q_i/\partial p_j$	$i \varepsilon C$ $j \varepsilon C$
λ_1 $\partial \tilde{\Phi}/\partial q_i$	$i \varepsilon C, G, Z$
λ_3 $\partial \sigma_o/\partial p_j$	$j \varepsilon C$
$\partial q_{ic}/\partial q_j$	$i \varepsilon X, Y$ $j \varepsilon G$
$\partial q_i/\partial q_j$	$i \varepsilon C$ $i \varepsilon G$
$\partial \sigma_o/\partial q_j$	$j \varepsilon G$
$\partial p_i/\partial q_i$	$i \varepsilon Z, V$
$\partial \Phi/\partial q_{ij}$	$i \varepsilon V, W_c$ $j \varepsilon G, Y_c, Z$
λ_{2i}	$i \varepsilon W_c$.

This suggests that the cooperative decision-maker's task of maximizing the profits of the cooperative's member patrons is a difficult one. In fact, it is doubtful that a cooperative of any complexity will be able to fully attain the objective of maximizing the profits of its member patrons.

Nevertheless, the optimality conditions presented here should be of value to the cooperative which is attempting to maximize its member patrons' profits even if it is not entirely successful in doing so. Further insight into these conditions is provided by considering the simplified models which follow. One result is immediate. Because of the interrelationships between the variables in this model, insistence on the principle that price must equal or exceed average total cost for

every service (service at cost, i.e., there cannot be loss-leaders) may conflict with the optimality conditions presented here, thus leading to a lower than optimal value of the objective function.

Single-Product Marketing Cooperative

In this model, the cooperative markets a product produced by single-product member and nonmember patrons. This product is used by the cooperative in the production of several outputs, each of which is sold outside the cooperative association. The cooperative does not supply its patrons with any inputs. All of these must be purchased from sources outside the cooperative association. In addition, the cooperative must purchase some of its inputs from sources outside the cooperative association.

Given these assumptions, only Kuhn-Tucker conditions 3.20 and 3.23 through 3.27 are relevant. Of these, the interpretations of all but 3.20 are similar to those corresponding to the general model. Condition 3.20a can be rewritten:

$$(p_x + q_{xc}\frac{\partial p_x}{\partial q_{xc}})\frac{\partial q_{xc}}{\partial p_x} - \sum_{i\epsilon Y_o} p_i \frac{\partial q_{ic}}{\partial p_x} - \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]$$

$$(p_x + q_x\frac{\partial p_x}{\partial q_x})\frac{\partial q_x}{\partial p_x} + \lambda_1 \frac{\partial \Phi}{\partial q_x}\frac{\partial q_x}{\partial p_x} - \lambda_3 \frac{\partial \sigma_o}{\partial p_x} = 0 \qquad (4.12)$$

where x represents the product marketed by the cooperative.

The term $p_x + q_x(\partial p_x/\partial q_{xc})$ represents the marginal revenue of the member patrons from product x. Thus, $[p_x + q_x(\partial p_x/\partial q_{xc})]\,\partial q_{xc}/\partial p_x$ represents the increase in total revenue of the member patrons arising

from the output shift induced by the variation in the price the cooperative offers for x (dq_x). The term $\sum_{i \epsilon Y_o} p_i(\partial q_{ic}/\partial p_x)$ represents the increase in the total cost of the member patrons due to the shifts in factor use which are induced by dq_x.

The term $p_x + q_x(\partial p_x/\partial q_x)$ represents the marginal factor cost to the cooperative of product x. Thus, $[p_x + q_x(\partial p_x/\partial q_x)]\, \partial q_x/\partial p_x$ represents the increase in the total cost of the cooperative from product x arising from the changes in the quantities supplied which are induced by dp_x.

The term $\lambda_1(\partial \Phi/\partial q_x)$ is interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of x used in production by the cooperative. This is equivalent to the marginal revenue product to the cooperative from product x multiplied by $s + (1-s)/(1+d_c)^{\tau}$. Thus the term $\lambda_1(\partial \Phi/\partial q_x)(\partial q_x/\partial p_x)$ can be interpreted as equivalent to the increase in the total revenue of the cooperative from use of product x arising from changes in the quantities supplied which are induced by dp_x, multiplied by $s + (1-s)/(1+d_c)^{\tau}$. The term $\lambda_3(\partial \sigma_o/\partial p_x)$ is interpreted as the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dp_x.

Thus, for a maximum, the sum of the increase in the total revenue of the member patrons from the output shift induced by dp_x; and the increase in the total revenue of the cooperative from use of produce x arising from changes in the quantities supplied which are induced by dp_x, discounted, must equal the sum of the increase in the total cost of

the member patrons due to the shifts in factor use which are induced by dp_x; the increase in the total cost of the cooperative from product x arising from the changes in the quantities supplied which are induced by dp_x, discounted; and the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business that is induced by dp_x.

In this model of a single-product marketing cooperative, $\partial\sigma_o/\partial p_x$, the change in the ratio of nonmember business to total business can be expressed:

$$\frac{\partial\sigma_o}{\partial p_x} = \frac{\sigma_o}{p_x}(\eta_o - \eta) \tag{4.13}$$

where η_o is defined as the elasticity of supply in the nonmember market:

$$\eta_o = \frac{\partial q_{xo}}{\partial p_x}\frac{p_x}{q_{xo}} \tag{4.14}$$

and η is defined as the elasticity of supply in the general market:

$$\eta = \frac{\partial q_x}{\partial p_x}\frac{p_x}{q_x}. \tag{4.15}$$

Because of member loyalty to the cooperative or the fact that member patrons expect to receive patronage refunds from the cooperative, the member market may, in general, be assumed to be less price-responsive than the nonmember market. Therefore, η_o may, in general, be assumed to be greater than η so that $(\eta_o - \eta)$ is greater than zero. Thus, if σ_o and p_x are positive, $\partial\sigma_o/\partial p_x$ will, in general, be positive and $\lambda_3(\partial\sigma_i/\partial p_x)$, the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total

business which is induced by dp_x, will be positive or equal to zero depending upon whether the value of λ_3 is positive or equal to zero.

No nonmember patrons

If it is assumed that the cooperative does not serve nonmember patrons, condition 4.12 can be simplified. Because the cooperative is a single-product cooperative, $\partial p_x/\partial q_x = 1/(\partial q_x/\partial p_x)$ by the inverse function rule of calculus. Simplifying 4.12 and multiplying it by $\partial p_x/\partial q_x$, it can be rewritten:

$$(1 - [s + \frac{(1-s)}{(1+d_c)^T}])(p_x + q_x \frac{\partial p_x}{\partial q_x}) - \sum_{i \in Y_o} p_i \frac{\partial q_i}{\partial q_x} + \lambda_1 \frac{\partial \Phi}{\partial q_x} = 0 \qquad (4.16)$$

for the case in which there are no nonmember patrons.

The term $p_x + q_x(\partial p_x/\partial q_x)$ represents the marginal revenue to the member patrons or the marginal factor cost to the cooperative of the product produced by the member patrons and marketed through the cooperative. As a marginal revenue, it represents an increase in the profits of the member patrons. However, as a marginal factor cost to the cooperative, multiplied by $s + (1-s_c)/(1+d)^T$, it represents a decrease in the amount of net savings available for distribution to the member patrons in patronage refunds. If the proportion of the allocated patronage refunds which are paid in the current period is equal to one (s=1) or if the cooperative sets its discount rate equal to zero ($d_c = 0$), the marginal revenue to the member patrons and the discounted marginal factor cost to the cooperative cancel and $p_x + q_x(\partial p_x/\partial q_x)$ vanishes from 4.16.

Thus, 4.16 is equivalent to stating that, for a maximum, the sum of the increase in the total revenue to the member patrons from the product

marketed by the cooperative and the discounted marginal revenue product of the product marketed by the cooperative should equal the sum of the increase in total cost to the member patrons due to the shifts in factor use which accompany an increase in the production of the product marketed by the cooperative and the discounted marginal factor cost to the cooperative of the product marketed by the cooperative.

If the proportion of the patronage refunds allocated in the current period which are paid in the current period is equal to one (s=1) or if the cooperative sets its discount rate equal to zero (d_c=0) and the product marketed through the cooperative is the only product produced by the member patrons, 4.16 is equivalent to stating that for a maximum, the marginal increase in the cost of the member patrons from producing the product should equal its marginal revenue product in the cooperative.

If, in addition, the typical member patron does not expect to re-receive any patronage refunds, from 2.7 its supply curve is its marginal cost curve above its average variable cost curve, represented by mc in the left panel of Figure 4.1. If this is the case, the supply curve facing the cooperative is the horizontal sum of the supply curves of the member patrons, represented by MC in the right panel of the figure. The optimum price in this example is p'_x, determined by the intersection of MRP and MC. The quantity supplied by the typical member patron will be q'_{xm}, and the total quantity supplied by the member patrons will be q'_x.[1]

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[1]According to Clark [11, pp. 38-39], total economic welfare is maximized at the quantity at which marginal cost is equal to average revenue. In this case, the average revenue to the member patrons from x is equivalent to its marginal revenue product in the cooperative. Thus, according to Clark's criterion, q'_x is the quantity at which total economic welfare is maximized.

If the typical member does expect to receive patronage refunds, its marginal cost curve will not be its supply curve. Instead, its supply curve, represented by s in Figure 4.1, will lie to the right of its marginal cost curve. The supply curve facing the cooperative, represented by S in the figure, will still be the horizontal sum of the supply curves of the member patrons but will lie to the right of MC.

If it is assumed that all member patrons have the same expectations and discount rates, each of their individual supply curves will be an equal distance below their marginal cost curves. If this is the case, the optimal price will no longer be p_x' for at this price q_x''' will be supplied and the marginal cost of the product will not equal its marginal revenue product in the cooperative. The optimal price will not be p_x'', determined by the intersection of MRP and S, for at this price q_x'' will be supplied and the marginal cost of the product again will not equal its marginal revenue product in the cooperative. The optimal price will be p_x''' for at this price q_x' will be supplied and the marginal cost of the product will equal its marginal revenue product in the cooperative.

The argument that the quantity at which the marginal cost of the product equals its marginal revenue product in the cooperative is the optimal quantity can be made in terms of producers' and consumers' surpluses. The producers' surplus can be defined as the difference between what the producers of the product (the member patrons) actually receive and what they would be willing to receive for a given quantity, a measure of the net benefit or profit they derive from selling the product. In Figure 4.1, the producers' surplus is represented by the area

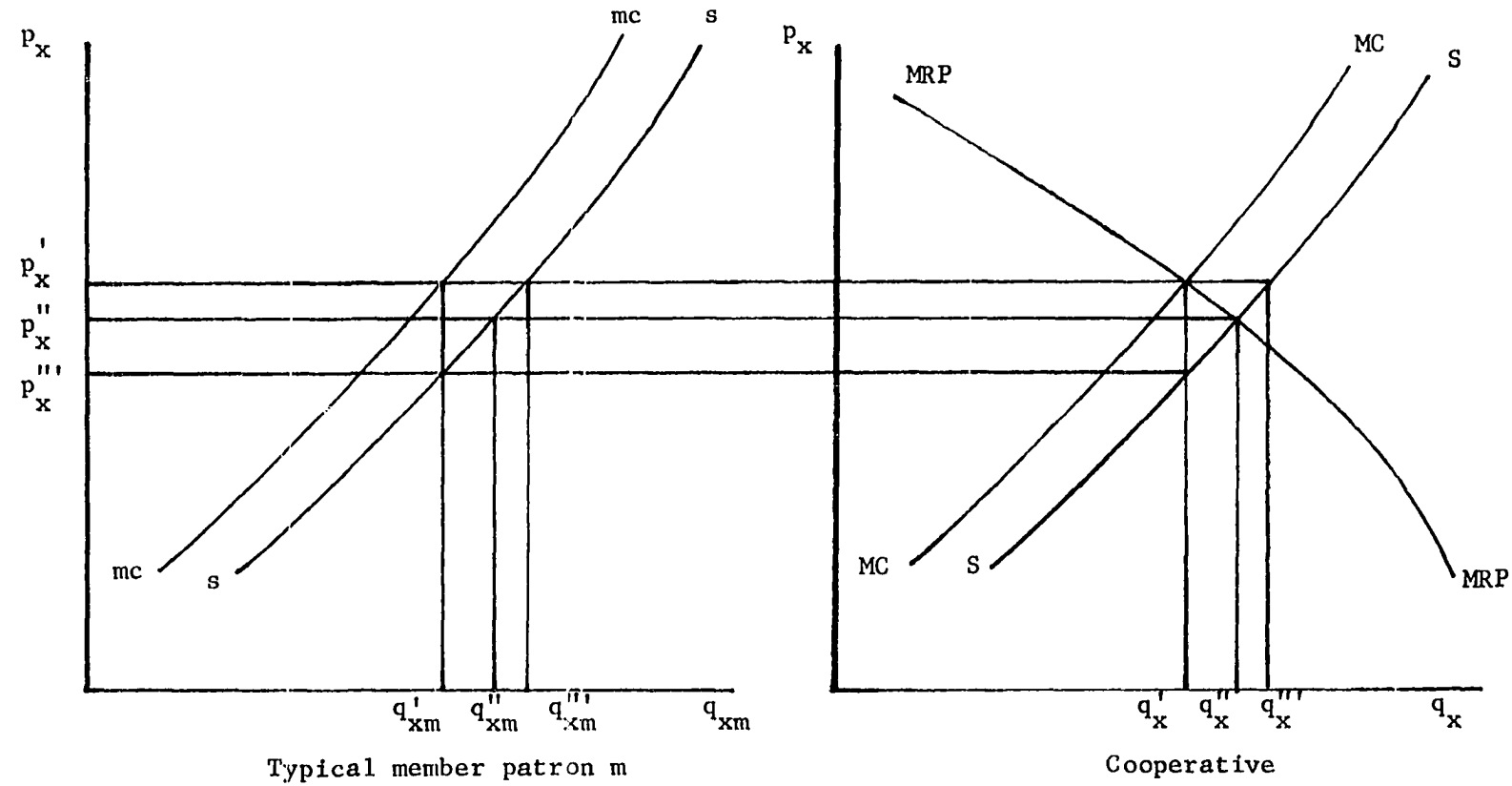

Figure 4.1. Single-product marketing cooperative.

below the horizontal line through the equilibrium price and above the marginal cost curve. The consumer's surplus can be defined as the difference between what the consumer of the product (the cooperative) would be willing to pay and what it actually pays for a given quantity, a measure of the net benefit or net savings it derives from purchasing the product. In the figure, the consumer's surplus is represented by the area above the horizontal line through the equilibrium price and below the marginal revenue product curve.

A proprietary firm might be interested in maximizing the consumer's surplus alone. This would be accomplished by operating at the point at which the marginal revenue product curve intersects the marginal factor cost curve instead of where it intersects the marginal cost or supply curve. However, the cooperative attempts to maximize the sum of the producers' and consumer's surpluses.

If the supply curve facing the cooperative is the marginal cost curve, the cooperative maximizes the profits of its member patrons by setting a price equal to the marginal revenue product of the product. Unless the marginal revenue product is equal to the average revenue product, this price by itself will not result in all of the producer surplus being distributed to the member patrons. A price equal to the average revenue product would by itself result in all of the producer surplus being distributed to the member patrons, but it would not result in a maximum. The cooperative, however, can set a price equal to the marginal revenue product and still distribute all of the producer surplus through the use of patronage refunds. This is an important point, and it takes

on more significance in the discussion of the Phillips model.

If all member patrons do not have the same expectations or discount rates, their individual supply curves will not be equidistant from their marginal cost curves. This suggests that the value of the maximum attained by the cooperative is dependent upon the value of the member patrons' expectations, parameters in the program. It also suggests that it will be difficult for the cooperative to identify the increase in the total cost of the member patrons due to an increase in quantity. Therefore, when attempting to make maximizing decisions, the cooperative may choose to assume that member patrons have identical expectations.

Single-Product Supply Cooperative

In this model, the cooperative supplies member and nonmember patrons with a single factor of production. This factor is used by the patrons in the production of several outputs, each of which is sold outside the cooperative association. The cooperative does not market any of the outputs for its patrons. In addition, the cooperative must purchase from outside the cooperative association the inputs which it uses in the production of the factor.

Given these assumptions, only Kuhn-Tucker conditions 3.21 and 3.24 through 3.28 are relevant. Of these, the interpretations of all but 3.21 are similar to those corresponding to the general model. Condition 3.21a can be rewritten:

$$- (p_y + q_{yc} \frac{\partial p_y}{\partial q_{yc}}) \frac{\partial q_{yc}}{\partial p_y} + \sum_{i \in X_o} p_i \frac{\partial q_{ic}}{\partial p_y} - \sum_{i \in Y_c} p_i \frac{\partial q_{ic}}{\partial p_y}$$

$$+ \left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right]\left(p_y + q_y \frac{\partial p_y}{\partial q_y}\right) \frac{\partial q_y}{\partial p_y} + \lambda_1 \frac{\partial \Phi}{\partial q_y} \frac{\partial q_y}{\partial p_y} - \lambda_3 \frac{\partial \sigma_o}{\partial p_y} = 0 \qquad (4.17)$$

where y represents the factor supplied by the cooperative.

The term $p_y + q_{yc}(\partial p_y/\partial q_{yc})$ represents the marginal factor cost to the member patrons of factor x. Thus, $[p_y + q_{yc}(\partial p_y/\partial q_{yc})]\, \partial q_{yc}/\partial p_y$ represents the increase in the total cost of the member patrons from shift in factor use induced by the variation in the price the cooperative charges for y. The term $\sum_{i \epsilon X_o} p_i(\partial q_{ic}/\partial p_y)$ represents the increase in total revenue to the member patrons arising from the output shifts which are induced by the variation in p_y (dp_y). The term $\sum_{i \epsilon Y_o} p_i(\partial q_{ic}/\partial p_y)$ represents the increase in total cost from products in set Y_o to the member patrons arising from the shifts in factor use which are induced by dp_y.

The term $p_y + q_y(\partial p_y/\partial q_y)$ represents the marginal revenue to the cooperative from factor y. Thus, $[p_y + q_y(\partial p_y/\partial q_y)]\, \partial q_y/\partial p_y$ represents the increase in the total revenue to the cooperative from factor y arising from the changes in the quantities supplied which are induced by dp_y.

The term $-\lambda_1(\partial \Phi/\partial q_y)$ is interpreted as the marginal variation in the profits of the member patrons arising from a change in the quantity of y produced by the cooperative. This is equivalent to the marginal cost to the cooperative of factor y multiplied by $s + (1-s)/(1+d_c)^{\tau}$. Thus, the term $-\lambda_1(\partial \Phi/\partial q_y)(\partial q_y/\partial p_y)$ can be interpreted as equivalent to the increase in the total cost to the cooperative of factor y arising from changes in the quantities demanded which are induced by dp_y, multiplied by

$s + (1-s)(1+d)^{T}$. The term $\lambda_3(\partial\sigma_o/\partial p_y)$ is interpreted as the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dp_y.

Thus, for a maximum, the sum of the increase in the total revenue of the member patrons from the output shifts induced by dp_y; and the increase in the total revenue of the cooperative from factor y arising from changes in the quantities demanded which are induced by dp_y, discounted, must equal the sum of the increase in the total cost of the member patrons due to the shifts in factor use which are induced by dp_y; the increase in the total cost of the cooperative from factor y arising from the changes in the quantities demanded which are induced by dp_y, discounted; and the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business that is induced by dp_y. This result is very similar to that derived earlier for a change in p_x in the model of a single-product marketing cooperative.

In this model of a single-product supply cooperative, $\partial\sigma_o/\partial p_y$, the change in the ratio of nonmember business to total business, can be expressed:

$$\frac{\partial\sigma_o}{\partial p_j} = \frac{\sigma_o}{p_y}(\varepsilon_o - \varepsilon) \tag{4.18}$$

where ε_o is defined as the elasticity of demand in the nonmember market:

$$\varepsilon_o = \frac{\partial q_{yo}}{\partial p_y}\frac{p_y}{q_{yo}} \tag{4.19}$$

and ε is defined as the elasticity of demand in the general market:

$$\epsilon = \frac{\partial q_y}{\partial p_y} \frac{p_y}{q_y} . \tag{4.20}$$

Again, as in the case of the single-product market cooperative, the member market may, in general, be assumed to be less price-responsive than the nonmember market. Therefore, ϵ_o may, in general be greater than ϵ so that $(\epsilon_o - \epsilon)$ is greater than zero. Therefore, if σ_o and p_x are positive, $\partial\sigma_o/\sigma p_y$ will, in general, be positive and $\lambda_3(\partial\sigma_o/\partial p_y)$, the marginal variation in the profits of the member patrons arising from the change in the ratio of nonmember business to total business which is induced by dp_y, will be positive or equal to zero depending upon whether the value of λ_3 is positive or equal to zero.

No nonmember patrons

If it is assumed that the cooperative does not serve nonmember patrons, condition 4.18 can be simplified. Because the cooperative is a single-product cooperative, $\partial p_y/\partial q_y = 1/(\partial q_y/\partial p_y)$ by the inverse function rule of calculus. Simplifying 4.18 and multiplying it by $\partial p_y/\partial q_y$, it can be rewritten:

$$- (1 - [s + \frac{(1-s)}{(1+d_c)^T}])(p_y + q_y \frac{\partial p_y}{\partial q_y} + \sum_{i \epsilon X_o} p_i \frac{\partial q_i}{\partial q_y}$$

$$- \sum_{i \epsilon Y_o} p_i \frac{\partial q_i}{\partial q_y} + \lambda_1 \frac{\partial \Phi}{\partial q_y} = 0 \tag{4.21}$$

for the cases in which there are no nonmember patrons.

The term $p_y + q_y(\partial p_y/\partial q_y)$ represents the marginal factor cost to the member patrons or the marginal revenue to the cooperative of the factor

produced by the cooperative and sold to the member patrons. As a marginal cost, it represents an increase in the costs of the member patrons. However, as a marginal revenue to the cooperative, multiplied by $s + (1-s)/(1+d_c)^{\tau}$, it represents an increase in the amount of net savings available for distribution to the member patrons in patronage refunds. If the proportion of the allocated patronage refunds which are paid in the current period is equal to one ($s = 1$) or if the cooperative sets its discount rate equal to zero ($d_c = 0$), the marginal factor cost to the member patrons and the discounted marginal revenue to the cooperative cancel and $p_y + q_y(\partial p_y/\partial q_y)$ vanishes from 4.21.

Thus, 4.21 is equivalent to stating that for a maximum, the sum of the increase in the total revenue due to the output shifts which accompany an increase in the use of the factor supplied by the cooperative and the discounted marginal revenue to the cooperative from the factor supplied by the cooperative should equal the increase in total cost to the member patrons due to the shifts in factor use and the discounted marginal cost to the cooperative of producing it.

If the proportion of the patronage refunds allocated in the current period which are paid in the current period is equal to one ($s=1$) or if the cooperative sets its discount rate equal to zero ($d_c=0$) and the factor supplied by the cooperative is the only variable factor used by the member patrons, 4.21 is equivalent to stating that for a maximum, the marginal cost of supplying the factor should equal the marginal increases in the revenue of the member patrons from using it.

If, in addition, the typical member patron does not expect to receive any patronage refunds, from 2.9 its demand curve is its marginal value product curve, represented by mvp in the left panel of Figure 4.2. If this is the case, the demand curve facing the cooperative is the horizontal sum of the demand curves of the member patrons, represented by MVP in the right panel of the figure. The optimum price in this example is p_y', determined by the intersection of MVP and MC. The quantity demanded by the typical member patron will be q_{ym}', and the total quantity demanded by the member patrons will be q_y'.

If the typical member does expect to receive patronage refunds, its marginal value product curve will not be its demand curve. Instead, its demand curve, represented by d in Figure 4.2, will lie to the right of the marginal value product curve. The demand curve facing the cooperative, represented by D in the figure, will still be the horizontal sum of the demand curves of the member patrons but will lie to the right of MVP.

If it is assumed that all member patrons have the same expectations, each of their individual demand curves will be an equal distance above their marginal value product curves. If this is the case, the optimal price will no longer be p_y' for at this price q_y''' will be demanded and the marginal value product of the factor will not equal its marginal cost. The optimal price will not be p_y'', determined by the intersection of MVP and D, for at this price q_y'' will be supplied and the marginal value product of the factor again will not equal its marginal cost. The optimal price will be p_y''' for at this price q_y' will be supplied and the marginal value product of the factor will equal its marginal cost.

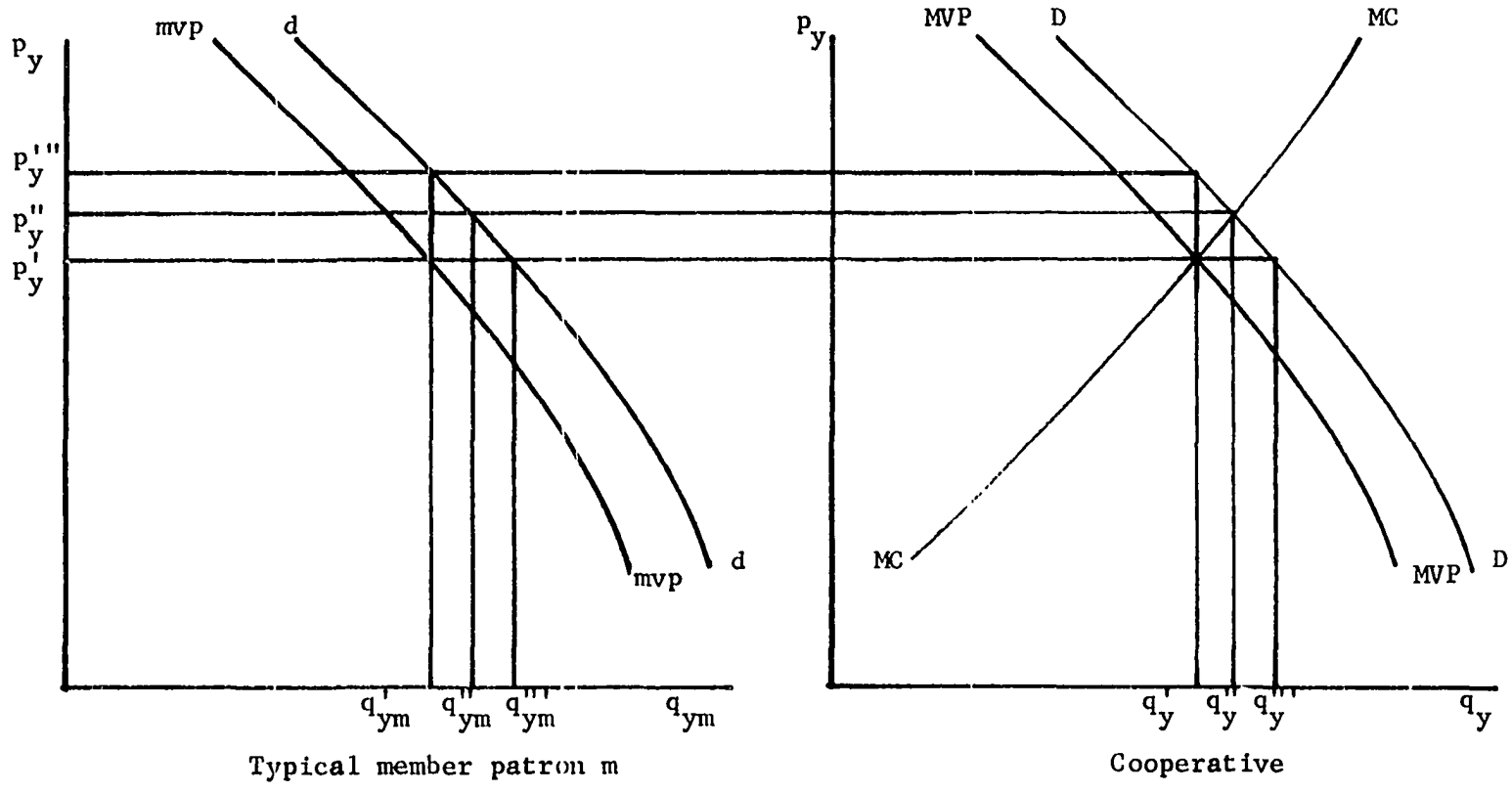

Figure 4.2. Single-product supply cooperative.

Similar to that concerning the model of the single-product marketing cooperative, the argument that the quantity at which the marginal value product of the factor equals its marginal cost is the optimal quantity can be made in terms of producer's and consumers' surpluses. With the exception that the producer is the cooperative and the consumers are the member patrons, the argument is identical to that used in the model of the single-product marketing cooperative.

This result is identical to that found by Enke [19] in his model of a consumer cooperative. He suggested that a consumer cooperative which took into account its consumers as owners as well as patrons should set the price it charges its members for a particular product equal to the marginal cost of producing it. A proprietary firm, serving as a supplier, would maximize its producer's surplus (profit) by operating at the point at which the marginal cost curve intersects the marginal revenue curve instead of the marginal value product curve or demand curve.

If the demand curve facing the cooperative is the marginal value product curve, the cooperative maximizes the profits of its member patrons by setting a price equal to the marginal cost of the factor. Unless the marginal cost is equal to the average cost, this price by itself will not result in all of the consumer surplus being distributed to the member patrons. A price equal to average cost would by itself result in all of the consumer surplus being distributed to the member patrons, but it would not result in a maximum. The cooperative, however, can set a price equal to the marginal cost and still distribute all of the consumer surplus through the use of patronage refunds.

Similar to the situation in the model of the single-product marketing cooperative, if all member patrons do not have the same expectations, their individual demand curves will not be equidistant from their marginal value product curves. In this case, it will be difficult for the cooperative to identify the increase in the total revenue due to an increase in quantity. Therefore, more support can be given to the suggestion that when attempting to make maximizing decisions, the cooperative may choose to assume that member patrons have identical expectations.

Phillips Model

In the Phillips model of a single-product marketing cooperative [44], member patrons produce a single raw product which is processed and then marketed by the cooperative. According to Phillips, each member patron maximizes its profits by producing the quantity at which the sum of its marginal cost and the marginal cost to the cooperative is equal to the marginal revenue from the processed product.

Aresvik [5] pointed out that the net savings that a member receives from the cooperative is usually the difference between the average revenue from the processed product and its average cost. He, therefore, argued that each member patron maximized its profit by producing the quantity at which the sum of its marginal cost and the average cost to the cooperative is equal to the average revenue of the processed product. Helmberger and Hoos [28, footnote 24, p. 285] indicated that their results were consistent with Aresvik's conclusions.

The model presented here can be made to correspond to the assumptions made in the Phillips model by assuming that the cooperative serves member patrons who produce a single-product (x), that the cooperative uses this product in the production of a single output (z) which it markets, that the proportion of the allocated patronage refunds which are paid in the current period is equal to one (s=1), and that the production of each unit of z requires exactly one unit of x.

Given these assumptions, condition 3.23a can be simplified to:

$$p_z + q_z \frac{\partial p_z}{\partial q_z} + \lambda_1 \frac{\partial \Phi}{\partial q_z} = 0. \tag{4.22}$$

The term $p_z + q_z(\partial p_z/\partial q_z)$ represents the marginal revenue to the cooperative from product z. The term $-\lambda_1(\partial\Phi/\partial q_z)$ is interpreted as the marginal cost to the cooperative association of producing product z. This consists of the marginal increase in the cost to the member patrons from supplying the raw product x used in the production of the processed product z plus the marginal cost to the cooperative of producing z. Thus, 4.22 is equivalent to stating that, for a maximum, the marginal increase in the cost to the member patrons from supplying the raw product x plus the marginal cost to the cooperative of producing z should equal the marginal revenue to the cooperative from producing z.

This result can be used to vindicate, in part, the conclusions arrived to by Phillips. If the typical member patron does not expect to receive any patronage refunds, from 2.7 its supply curve is its marginal cost curve above its average variable cost curve, represented by mc_x in the left panel of Figure 4.3. If this is the case, the supply curve facing

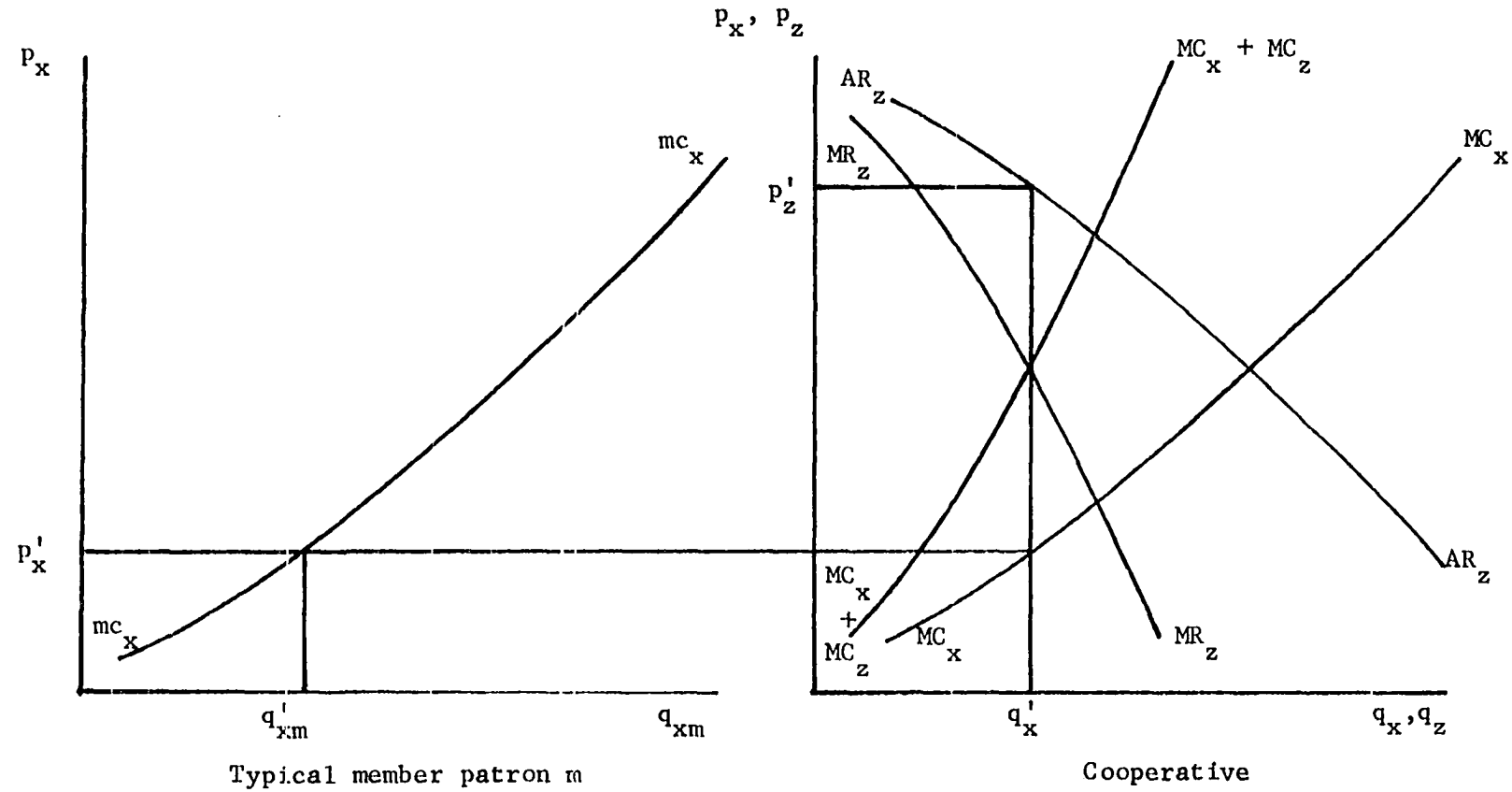

Figure 4.3. Phillips model of a marketing cooperative.

the cooperative is the horizontal sum of the supply curves of the member patrons, represented by MC_x in the right panel of the figure. The optimum price the cooperative should offer for the raw product x is p'_x, determined by the intersection of the cooperative's marginal revenue curve MR and and $MC_x + MC_z$, the curve which represents the sum of the marginal cost to the member patrons of producing x (MC_x) and the marginal cost to the cooperative of processing it (MC_z) The quantity of x supplied by the typical member patron will be q'_{xm}, and the total quantity supplied by the member patrons will be q'_x. The price the cooperative receives for the processed product z is determined from the average revenue curve AR_z and will be p'_z.

Thus, in this example, Phillips' condition that, for a maximum, the sum of the marginal cost to the member patron and the marginal cost to the cooperative should be equal to the marginal revenue from the processed product holds. However, it is important to point out that, in this model, it is the decisions of the cooperative, not of the member patrons, which ensure that a maximum is obtained. If it is construed that the member patrons make all decisions, Trifon's [51] criticisms (mentioned in the literature review) are still valid.

If all member patrons are assumed to have the same expectations of the per-unit patronage refund, their individual supply curves will be equidistant from their marginal cost curves as in the model of the single product marketing cooperative. If this is the case, Phillips' condition again holds for a maximum. However, if it is assumed that all member patrons do not have the same expectations, their individual supply curves

will not be equidistant from their marginal cost curves and Phillips' condition cannot be shown to always hold at a maximum.

Aresvik's condition that, for a maximum, the sum of the marginal cost to the member patron and the average cost to the cooperative should be equal to the average revenue from the processed product does not hold for the cases of zero or identical member patron expectations. Aresvik's contention that this condition holds at a maximum is based on the fact that the net savings that a member receives from the cooperative is usually the difference between the average revenue from the processed product and its average cost.

This is true, and if the cooperative must distribute its net savings to its member patrons solely through the price it offers them, it will produce the quantity of product z at which the sum of the marginal cost to the member patron and the average cost to the cooperative equals the average revenue from the product. At this quantity, the sum of the marginal costs will be greater and the marginal revenue will be lower than at the maximum. Therefore, profits will be lower than at the maximum.

However, if the cooperative is able to utilize patronage refunds to distribute its net savings, it will be able to price in such a manner that the maximum quantity of z is produced. The sum of the price and the per-unit patronage refund, not the price alone, will then be equal to the difference between the average revenue from the processed product and its average cost. In both the Aresvik discussion of the Phillips

model and the Helmberger and Hoos model, prices alone are used to distribute the net savings of the cooperative to its member patrons.[1] It is also true that in the original Phillips model, no mention is made of patronage refunds. Thus, although Phillips suggested the correct optimality condition, his model is not satisfactory in that it does not provide a mechanism by which the cooperative can distribute its net savings to its member patrons.

In the Phillips model of a single-product supply cooperative, member patrons purchase a single raw product from the cooperative. This product is processed and then marketed by the individual member patrons. The member patrons are single-product firms in that this processed product is their only output. Again, according to Phillips, each member patron maximizes its profits by producing the quantity at which the sum of its marginal cost and the marginal cost to the cooperative is equal to the marginal revenue from the processed product.

The model presented here can be made to correspond to the assumptions made in the Phillips model by assuming that the cooperative produces a

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[1]The Helmberger and Hoos model is actually a generalization of the Phillips-Aresvik model in which the amount of the raw product which is required to produce a unit of the processed product is not fixed. Because prices alone are used to distribute the net savings of the cooperative to its member patrons in the Helmberger and Hoos model, it is subject to the same criticisms which can be applied to the Aresvik discussion of the Phillips model. In the Helmberger and Hoos model, the price of the raw product equals the difference between the price of the processed product and its average total cost, multiplied by the ratio of the quantity of the processed product to the quantity of the raw product used (which is equal to one in the Aresvik example). If the cooperative is able to utilize patronage refunds to distribute its net savings, the price of the raw product should equal its marginal product. The Hardie model is not subject to this criticism because of the special assumptions of the linear programming model.

single product (y), that the member patrons use this product in the production of a single output (x) which they market, that the proportion of the allocated patronage refunds which are paid in the current period is equal to one (s=1), and that the production of each unit of x requires exactly one unit of y.

Given these assumptions, condition 3.21a can be simplified to:

$$p_x - \sum_{i \epsilon Y_o} p_i \frac{\partial q_i}{\partial q_x} + \lambda_1 \frac{\partial \Phi}{\partial q_y} = 0. \qquad (4.23)$$

The term p_x represents the marginal revenue to the member patrons from product x. Because it is assumed in this study that member patrons are price-takers, the marginal revenue to the member patrons is equal to the price. The term $\sum_{i \epsilon Y_o} p_i(\partial q_i/\partial q_x)$ can be interpreted as the marginal increase in the cost to the member patrons from producing x. Finally, the term $-\lambda_1(\partial \Phi/\partial q_y)$ can be interpreted as the marginal cost to the cooperative of producing y. Thus, 4.23 is equivalent to stating that, for a maximum, the marginal increase in the cost to the member patrons from producing x plus the marginal cost to the cooperative of producing y should equal the marginal revenue to the member patrons from product x.

It can be shown that if the member patrons do not expect to receive any patronage refunds or if it is assumed that all member patrons have the same expectations of the per-unit patronage refund, Phillips' condition will hold for a maximum. However, the same criticisms which were made of the Phillips model of a single-product marketing cooperative can be made of the Phillips model of a single-product supply cooperative. It

is the cooperative, not member patrons, which must ensure that a maximum is obtained. In addition this model does not provide a mechanism by which the cooperative can distribute its net savings to its member patrons.

CHAPTER V. FUTURE EFFECTS

Model

Expression 2.3 provides a mechanism by which decisions made during the current period may affect the profits of the member patrons in future periods. The mechanics of these effects are represented by 2.56 and 2.57. Decisions which affect the net savings of a department within the cooperative affect the patronage refunds on the products in that department and, therefore, may affect the decisions made by member patrons in future periods through the effect the current patronage refunds have on the member patron's expectations of future refunds.

In the model presented in this chapter, the effects current decisions have on the profits of the member patrons in future periods are considered. The cooperative decision-maker is assumed to maximize the total discounted profits of its member patrons over the time horizon of the cooperative:

$$\hat{\Pi} = \sum_{t=0}^{T} \sum_{m=1}^{M} \pi_{mt}/(1+d_f)^t \qquad (5.1)$$

where π_{mt} is the profit of the m-th member firm in the t-th period and d_f is the discount rate used to determine the present value of future profits. This rate is set by the cooperative decision-maker and may be different than d_c. The time horizon T is defined as the number of periods in which decisions made in the current period have an effect.

It should be noted that the profits of the member patrons in future periods (t=1, 2, ..., T) are expected profits. The cooperative must make decisions based on a set of expected prices and on expected forms of the

production functions and the demand and supply functions. The profits of the member patrons in the current period (t=0) are, however, actual profits. Whereas, the member patrons do not know the per-unit patronage refunds for the current period and, therefore, must maximize expected profits, the cooperative determines the per-unit patronage refunds and can, therefore, maximize actual profits in the current period.

The objective function 5.1 which the cooperative is assumed to maximize is subject to several constraints. There is a production function similar to 3.2 for each time period t. There is also a set of fixed factor constraints similar to 3.16 and a nonmember-business constraint 3.17 for each time period t. Finally, 3.15 is included as a constraint. It ensures that the total of the amounts of the indirect departmental costs allocated must equal the total indirect departmental cost.

With the problem as stated, the Lagrangian function can be expressed:

$$\Gamma = \sum_{t=0}^{T} \left(\sum_{i \in X} p_{it}\, q_{ict} - \sum_{i \in Y} p_{it}\, q_{ict} + \left[s + \frac{(1-s)}{(1+d_c)^{\tau}} \right] NS_t \right) / (1+d_f)^t$$

$$+ \sum_{t=0}^{T} \lambda_{1t} \cdot \Phi_t + \sum_{t=0}^{T} \lambda_{2it} \left(q_{i0t} - \sum_{j \in C,Z} q_{ijt} \right) + \sum_{t=0}^{T} \lambda_{3t} [\sigma - \sigma_{ot}]$$

$$+ \lambda_4 \left(C_I - \sum_{k} C_{Ik} \right) \qquad (5.2)$$

where λ_{1t} is the Lagrange multiplier corresponding to the production function for the t-th period, the λ_{2it} are the Lagrange multipliers corresponding to the fixed-factor constraints for the t-th period, λ_{3t}

is the Lagrange multiplier corresponding to the nonmember-business constraint for the t-th period, and λ_4 is the Lagrange multiplier corresponding to the indirect-cost-allocation constraint. In this function as in the rest of this chapter, a symbol with a t subscript denotes a value in the t-th period. Absence of a t subscript denotes a value in the current period (in which t = 0).

As with the model presented in Chapters III and IV, the prices that the cooperative sets for the products in set C, the quantities of the products in sets G and Z it produces, and the quantities of each of the variable inputs in set V and of each of the fixed factors in set W_c which the cooperative uses in the production of each of the products in sets G, Y_c, and Z are instruments. In addition, because the decisions made by member patrons in future periods may be affected by current per-unit patronage refunds, the cost allocations of the cooperative, determined in the second stage of the decision-making prices, are also instruments in this model.

Given these instruments, the Kuhn-Tucker conditions for the problem represented by 5.2 are as follows. Except for 5.16, the numbers in parentheses represent the terms in the corresponding Kuhn-Tucker conditions for the model in which future effects are not considered. 5.16 is presented in order, following the Kuhn-Tucker conditions.

for all $j \epsilon X_c$:

$$\frac{\partial \Gamma}{\partial p_j} = (3.20a) + (5.16) \leq 0 \qquad (5.3a)$$

$$\frac{\partial \Gamma}{\partial p_j} \cdot p_j = 0 \qquad (5.3b)$$

$$p_j \geq 0 \tag{5.3c}$$

for all $j \epsilon Y_c$:

$$\frac{\partial \Gamma}{\partial p_j} = (3.21a) + (5.16) \leq 0 \tag{5.4a}$$

$$\frac{\partial \Gamma}{\partial p_j} \cdot p_j = 0 \tag{5.4b}$$

$$p_j \geq 0 \tag{5.4c}$$

for all $j \epsilon G$:

$$\frac{\partial \Gamma}{\partial q_j} = (3.22a) \leq 0 \tag{5.5a}$$

$$\frac{\partial \Gamma}{\partial q_j} \cdot q_j = 0 \tag{5.5b}$$

$$q_j \geq 0 \tag{5.5c}$$

for all $j \epsilon Z$:

$$\frac{\partial \Gamma}{\partial q_j} = (3.23a) + (5.16) \leq 0 \tag{5.6a}$$

$$\frac{\partial \Gamma}{\partial q_j} \cdot q_j = 0 \tag{5.6b}$$

$$q_j \geq 0 \tag{5.6c}$$

for all $i \epsilon V$; $j \epsilon Y_c, Z$:

$$\frac{\partial \Gamma}{\partial q_{ij}} = (3.24a) + (5.16) \leq 0 \tag{5.7a}$$

$$\frac{\partial \Gamma}{\partial q_{ij}} \cdot q_{ij} = 0 \tag{5.7b}$$

$$q_{ij} \geq 0 \tag{5.7c}$$

for all $i \epsilon V$; $j \epsilon G$:

$$\frac{\partial \Gamma}{\partial q_{ij}} = (3.24a) \leq 0 \tag{5.8a}$$

$$\frac{\partial \Gamma}{\partial q_{ij}} \cdot q_{ij} = 0 \tag{5.8b}$$

$$q_{ij} \geq 0 \tag{5.8c}$$

for all $i \epsilon W_c$; $j \epsilon Y_c$, Z:

$$\frac{\partial \Gamma}{\partial q_{ij}} = (3.25a) + (5.16) \leq 0 \tag{5.9a}$$

$$\frac{\partial \Gamma}{\partial q_{ij}} \cdot q_{ij} = 0 \tag{5.9b}$$

$$q_{ij} \geq 0 \tag{5.9c}$$

for all $i \epsilon W_c$; $j \epsilon G$:

$$\frac{\partial \Gamma}{\partial q_{ij}} = (3.25a) \tag{5.10a}$$

$$\frac{\partial \Gamma}{\partial q_{ij}} \cdot q_{ij} = 0 \tag{5.10b}$$

$$q_{ij} \geq 0 \tag{5.10c}$$

for the k-th department:

$$\frac{\partial \Gamma}{\partial C_{Ik}} = -\left[s + \frac{(1-s)}{(1+d_c)^{\tau}}\right] + (5.16) - \lambda_4 \leq 0 \tag{5.11a}$$

$$\frac{\partial \Gamma}{\partial C_{Ik}} \cdot C_{Ik} = 0 \tag{5.11b}$$

$$C_{Ik} \geq 0 \tag{5.11c}$$

for λ_{1t}:

$$\frac{\partial \Gamma}{\partial \lambda_{1t}} = \Phi_t = 0 \tag{5.12}$$

for λ_{2it}, $i \epsilon W_c$:

$$\frac{\partial \Gamma}{\partial \lambda_{2it}} = q_{i0t} - \sum_{j \epsilon C,Z} q_{ijt} \geq 0 \tag{5.13a}$$

$$\frac{\partial \Gamma}{\partial \lambda_{2it}} \cdot \lambda_{2it} = 0 \tag{5.13b}$$

$$\lambda_{2it} \geq 0 \tag{5.13c}$$

for λ_{3t}:

$$\frac{\partial \Gamma}{\partial \lambda_{3t}} = \sigma - \sigma_{ot} \geq 0 \tag{5.14a}$$

$$\frac{\partial \Gamma}{\partial \lambda_{3t}} \cdot \lambda_{3t} = 0 \tag{5.14b}$$

$$\lambda_{3t} \geq 0 \tag{5.14c}$$

for λ_4:

$$\frac{\partial \Gamma}{\partial \lambda_4} - C_I - \sum_k C_{Ik} = 0 \tag{5.15}$$

where:

$$(5.16) = \sum_{t>0}^{T} \sum_{i \epsilon S} \sum_{m=1}^{M} \left(\sum_{g \epsilon X} p_{gt} \frac{\partial q_{gmt}}{\partial r^*_{imt}} - \sum_{g \epsilon Y} p_{gt} \frac{\partial q_{gmt}}{\partial r^*_{imt}} \right)$$

$$\frac{\partial r^*_{imt}}{\partial r_i} \frac{\partial r_i}{\partial I} / (1+d_f)^t - \left[s + \frac{(1-s)}{(1+d_c)^T} \right] \sum_{t>0}^{T} \sum_{i \epsilon S} \sum_{m=1}^{M} \left(\sum_{g \epsilon X_c} p_{gt} \frac{\partial q_{gmt}}{\partial r^*_{imt}} \right.$$

$$\left. - \sum_{g \epsilon Y} p_{gt} \frac{\partial q_{gmt}}{\partial r^*_{imt}} \right) \frac{\partial r^*_{imt}}{\partial r_i} \frac{\partial r_i}{\partial I} / (1+d_f)^t + \sum_{t>0}^{T} \sum_{i \epsilon S} \sum_{m=1}^{M} \sum_{g \epsilon C}$$

$$\lambda_{1t} \frac{\partial \Phi_t}{\partial q_{gt}} \frac{\partial q_{gmt}}{\partial r^*_{imt}} \frac{\partial r^*_{imt}}{\partial r_i} \frac{\partial r_i}{\partial I} - \sum_{t>0} \sum_{i \epsilon C} \lambda_{3t} \frac{\partial \sigma_{ot}}{\partial r_i} \frac{\partial r_i}{\partial I}$$

in which S represents the subset of products in set C, the per-unit patronage refunds of which are affected by the instrument which is represented by I.

Because a change in the price the cooperative sets for any product in set C can affect the quantity supplied or demanded of any product in set C (see 2.43), S=C for $I=p_j$, $j \epsilon C$. All other instruments affect only the per-unit patronage refunds for the products in a single department, the net savings of which is affected by the instrument. Thus, $S=X_j$, where X_j is the subset of products in set X purchased by the department which produces product j, for $I=q_j$, $j \epsilon Z$. $S=D_j$, where D_j is the subset of products in set X_c purchased by the department which produces product j if it is a marketing department and is the subset of products in set Y_c produced by the department if it is a supply department, for q_{ij}, $i \epsilon V$, $j \epsilon Y_c$, Z. Finally, $S=D_k$ where D_k is the subset of products in set X_c purchased by the k-th department if it is a marketing department and is the subset of products in set Y_c purchased by the k-th department if it is a supply department, for $I=C_{Ik}$.

The symbol r^*_{imt} in 5.16 represents the m-th member patron's expected per-unit patronage refund for the i-th product in the t-th period. The partial derivative $\partial q_{gmt}/\partial r^*_{imt}$ is given for the typical member patron by 2.56, in which k=g and l=i. The partial derivative $\partial r^*_{imt}/\partial r_i$ is determined by 2.3. Because nonmember patrons do not receive patronage refunds, they are not involved in the future effects of current decisions.

The partial derivative $\partial r_i/\partial I$ represents the effect of a change in the instrument I on the i-th per-unit patronage refund. It can be determined by 3.11, using information from 3.9 and from 3.12. Because the expansion of $\partial r_i/\partial I$ provides no further economic insight into the effect, it is not presented here.

Analysis

In 5.16, the term $\sum_{g\epsilon X} p_{gt}(\partial q_{gmt}/\partial r^*_{imt})$ represents the marginal variation in the total revenue of the m-th member patron in period t ($t > 0$) from the products in set X arising from output shifts which are induced by a variation in the member firms expectation of the per-unit patronage refund for product i (dr^*_{imt}). The term $\sum_{g\epsilon Y} p_{gt}(\partial q_{gmt}/\partial r^*_{imt})$ represents the marginal variation in the total cost of the m-th member patron in period t from the products in set Y arising from shifts in factor use which are induced by dr^*_{imt}.

Again, letting s' represent $s + (1-s)/(1+d_c)^{\tau}$, the term $s' \sum_{g\epsilon X_c} p_{gt}$ $(\partial q_{gmt}/\partial r^*_{imt})$ represents the marginal variation in total collective costs in period t from the products in set X_c arising from changes in the quantities supplied by the m-th member patron which are induced by dr^*_{imt}. The term $s' \sum_{g\epsilon Y_c} p_{gt}(\partial q_{gmt}/\partial r^*_{imt})$ represents the marginal variation in total collective revenues in period t from the produces in set Y_c arising from changes in the quantities demanded by the m-th member patron which are induced by dr^*_{imt}:

The term $-\lambda_{1t}(\partial \Phi^*_t/\partial q_{gt})$, $g \epsilon Y_c$, can be interpreted as the

The term $-\lambda_{1t}(\partial\Phi_t/\partial q_{gt})$, $g \in Y_c$, can be interpreted as the marginal variation in the discounted profits of the member patrons in the t-th period arising from a change in the quantity of output g in set Y_c produced in the t-th period, and $\lambda_{1t}(\partial\Phi/\partial q_{gt})$, $g \in X_c$, can be interpreted as the marginal variation in the discounted profits of the member patrons in the t-th period arising from a change in the quantity of input g in set X_c used in production by the cooperative in the t-th period.

The term $\lambda_{3t}(\partial\sigma_{ot}/\partial r_i)$, $i \in C$, can be interpreted as the marginal variation in the discounted profits of the member patrons in the t-th period from the change in the ratio of nonmember business to total business in the t-th period which is induced by a variation in the current per-unit patronage refund for the i-th product in set C. The partial derivative $\partial\sigma_{ot}/\partial r_i$ can be determined by applying the quotient rule of calculus on 3.18, in which t=0 and i=g. Unfortunately, this expansion provides little economic insight into this effect and is, therefore, not presented.

Given these interpretations, 5.16 can be interpreted as the net increase (or decrease) in the discounted profits of the member patrons in future periods arising from an increase in the instrument I. In each of the conditions, 5.3, 5.4, 5.5, 5.7, and 5.9, the terms represented by 5.16 accompany the terms which appear in the corresponding conditions for the model in which future effects are not considered.

Stated briefly, if the future effects are ignored, as they are in the current-effects model of the previous chapters, each of these conditions is equivalent to stating that, for a maximum, the increase in the

total current revenue of the member patrons (including the increase in the total collective revenues of the member patrons) resulting from a change in the value of the instrument must equal the increase in the total current cost of the member patrons (including the increase in the total collective costs of the member patrons) resulting from a change in the value of the instrument. If the future effects are considered, the terms represented by 5.16 appear in the conditions. Thus, if the net increase in the discounted profits of the member patrons in future periods arising from an increase in the instrument I is positive, the optimal value of the instrument will be greater than if only the current effects are considered.

Conditions 5.5, 5.8, and 5.9 concern the optimal level of the public goods and the factors used by the cooperative to produce them. Because it is not assumed that the current level of the public goods affects future production and because the cost of providing the public goods is charged to indirect departmental costs instead of to the departments, there are no future effects accompanying the provision of the public goods. Thus, conditions 5.5, 5.8, and 5.9 are identical to conditions 3.22, 3.24, and 3.25 for the current-effects model.

Condition 5.11 concerns the allocation of the indirect departmental costs to the departments and does not appear in the current-effects model. This is because in the second stage of maximization, the total indirect departmental cost C_I is a constant. The distribution of patronage refunds will be contingent upon the allocation of these costs, but the level of the total current profits of the member patrons will not

because current patronage refunds do not affect current behavior. However, when future effects are considered, 5.11 becomes important because current patronage refunds do affect future behavior.

The Lagrange multiplier λ_4 indicates how much the profits of the member patrons would increase with a one-unit increase in the level of C_I, the total indirect departmental cost. A one-unit increase in C_I would decrease the amount of the net savings of the cooperative available for distribution as patronage refunds. Thus, λ_4 is equal to $- s + (1-s)/(1+d_c)^T$ and condition 5.11a is equivalent to 5.16.

Hence, 5.11 is equivalent to stating that, for a maximum, if a positive allocation of indirect departmental costs is made to the k-th department, the net increase in the discounted profits of the member patrons in future periods arising from an increase in the level of indirect departmental costs allocated to the department should be equal to zero. In other words, the sum of the discounted future costs in 5.16 resulting from the department's allocation should equal the sum of the discounted future revenues resulting from it.

Conditions 5.12 through 5.15 are simply restatements of the constraints.

As apparent from 5.16, the effects of current decisions on future profits are complex. The difficulty of considering the future effects of current decisions is complicated by the amount of information concerning the future which the cooperative decision-maker must know in order to evaluate these effects. Among the values which the cooperative decision-

maker must know for each of the future periods within the time horizon (t=1, ..., T) are:

p_{gt}	$g \epsilon X, Y$
$\partial q_{gmt} / \partial r^*_{imt}$	$g \epsilon X, Y$ $m=1, \ldots, M$ $i \epsilon C$
$\partial r^*_{imt} / \partial r_i$	$i \epsilon C$ $m=1, \ldots, M$
λ_{1t}	
$\partial \Phi_t / \partial q_{gt}$	$g \epsilon C$
λ_{3t}	
$\partial \sigma_{ot} / \partial r_i$	$i \epsilon C$.

Many of these values are contingent upon future decisions by the cooperative. Some, such as the future prices of products in sets X_o and Y_o are even determined outside of the cooperative association.

This suggests that if the cooperative is to take future effects into consideration during its decision-making process, it must develop forecasting techniques capable of estimating some of the future values. However, because of the volume of information concerning the future which is necessary, it is likely that the cooperative decision-maker will have only a rough idea of how his decisions will affect future profits and that the time horizon of the cooperative cannot extend very far into the future.

CHAPTER VI. SUMMARY AND CONCLUSIONS

Summary

Some of the problems of the cooperative association were discussed, and related literature was reviewed in an effort to see how well it provided solutions. The purpose of this study was established as that of developing a short-run model of the cooperative association which could be used to analyze problems not discussed in the literature.

In particular, an attempt was made to develop a normative-prescriptive model of a multi-product marketing and supply cooperative which served both member and nonmember patrons. Using this model, analyses of the decisions of the cooperative decision maker on prices, patronage refunds, allocation of joint fixed costs, and determination of the optimal level of cooperative-provided public goods were performed.

Development of the model began with the construction of a non-linear programming sub-model of a typical multi-product member patron. The typical member patron was assumed to maximize its expected profits, including the present value of its expected patronage refunds. Expected per-unit patronage refunds were assumed to be functions of the actual per-unit refunds in past periods. The typical member patron purchased inputs from the cooperative and marketed some of its outputs through the cooperative, but was not required to deal exclusively with the cooperative. In addition, the production of the member patron was augmented by the provision of public goods by the cooperative.

A similar sub-model of a typical nonmember patron was also developed. The primary difference between this sub-model and that of the typical member patron was that the nonmember patron did not receive patronage refunds.

From the optimality conditions determined for these two sub-models, individual output supply and input demand functions were derived for the patrons. The prices set by the cooperative and the markets outside the cooperative association and the levels of the cooperative-provided public goods were arguments in all of these functions. In addition, the expected per-unit patronage refunds were arguments in the functions of the typical member patron. By horizontally summing these individual supply and demand functions across all member and across all nonmember patrons, aggregate functions were determined.

The cooperative decision-maker was assumed to maximize the total profits of its member patrons. This was accomplished in two stages. In the first stage, the decision-maker determined the optimal prices for the products it marketed and supplied and the optimal level of public goods it provided. In the second stage, at the end of its accounting period when its costs and net savings are known, it determined patronage refunds.

The optimality conditions for the general model of the cooperative were analyzed. Simplified models, including that of a single-product marketing cooperative, that of a single-product supply cooperative, and the Phillips model, were also analyzed and were compared to the literature.

Finally, the general model of the cooperative was extended to enable consideration of the effects on the member patrons' future profits of current decisions. The cooperative decision-maker was assumed to maximize the total discounted profits of the member patrons over the time horizon of the cooperative. In this model, the allocation of indirect departmental costs to the departments of the cooperative became an instrument. The optimality conditions for this model were analyzed and compared to those of the previous one.

Conclusions

The principal conclusion determined in this study is that the task of the cooperative decision-maker is a difficult one. The optimality conditions derived for the cooperative in this study are complex. In addition to the revenue and cost terms which are associated with the production and marketing activities of the cooperative and which have analogues in the optimality conditions for the firm, the optimality conditions for the cooperative also include revenue and cost terms corresponding to the market activities of the member patrons. In addition, they include terms representing variations in the profits of the member patrons arising from changes in the ratio of nonmember business to total business. Not only are the optimality conditions which were derived for the cooperative in this study complex, but there is a great amount of information which is necessary to evaluate them. It is doubtful that a cooperative of any complexity will be able to fully attain the objective of maximizing its member patrons' profits.

This point is especially relevant when the effects on the member patrons' future profits of current decisions are considered. When the future effects are considered, the optimality conditions become more complex and there is a greater amount of information which is necessary to evaluate them. Much of this information is contingent upon the future decisions of the cooperative. In fact, some of it is determined outside of the cooperative association.

This suggests that if the cooperative is to take future effects into consideration during its decision-making process, it must develop forecasting techniques capable of estimating some of this information. Still, it is likely that the cooperative decision-maker will have only a rough idea of how his decisions will affect future profits and that the time horizon of the cooperative cannot extend very far into the future.

Nevertheless, many of the results of this study should be useful. The optimality conditions presented here should be of value to the cooperative which is attempting to maximize the profits of its member patrons even if it is not entirely successful in doing so. From the outset, the model developed in this study was never intended to be a positive one but a normative-prescriptive one--one which would provide rules of behavior by which cooperatives might strive to optimize their member patrons' profits.

The model presented in this study was used to clarify several points in the theory surrounding cooperatives. A distinction was drawn between the marginal cost found in a model of a multi-product firm and the marginal variable cost found in a model of a single-product firm. The

difference is that the former must include the marginal opportunity cost of using fixed factors of production.

It was demonstrated that under the assumptions of the Phillips model, his conclusion that, for a maximum, the sum of the marginal cost to the member patron and the marginal cost to the cooperative should be equal to the marginal revenue from the processed product may indeed be true. However, it was pointed out that it is the decisions of the cooperative, not of the member patrons, which ensure that a maximum is obtained. It was also pointed out that the Phillips model does not provide a mechanism by which the cooperative can distribute its net savings to its member patrons.

In a related way, it was shown that if a single-product cooperative must distribute its net savings to its member patrons solely through the price it sets, the profits of the member patrons may be lower than if the cooperative is able to utilize patronage refunds to distribute its net savings. The use of patronage refunds ensures that the net savings of the cooperative can be distributed to its member patrons while prices are used to fulfill the optimality conditions.

Finally, it was demonstrated that the allocation of joint costs can be used as an instrument if the future behavior of member patrons is assumed to be influenced by the level of current per-unit patronage refunds.

Further Research

As mentioned earlier in this study, there is no general agreement on what the objective or objectives of the cooperative are or should be. It would be interesting to carry out the type of analysis performed here for different objective functions and to compare the results. Candidates might include maximization of a weighted sum of the member patrons' profits, the net savings (or patronage refunds) of the cooperative, or a multiple-argument objective function. Arguments in a multiple-argument function might include the quantities of various products sold or marketed through the cooperative.

It was also indicated earlier that there were two types of public goods which might be provided by the cooperative. There are those which affect production and there are those which affect price. Only the former was analyzed here. Although Ladd [35] has done some work with the latter, it would be interesting to do more. In particular, it would be interesting to analyze the effects price-augmenting, cooperative-provided public goods might have on the prices the member patrons might receive from markets outside the cooperative association.

Finally, it might be interesting to examine alternative methods of determining patronage refunds when there is a department in the cooperative which consists of products which are sold to patrons and which rely on products purchased from patrons as major inputs. Such an analysis might involve both questions of efficiency and equity.

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APPENDIX A. LIST OF SYMBOLS

The following is a list of symbols used in the models presented in this study. The symbols are listed in alphabetical order, those in the Roman alphabet preceding those in the Greek alphabet. If a number follows the explanation of a particular symbol, it refers to the equation in which the corresponding term is defined. The existence of a t subscript denotes a value in the t-th time period. The absence of a t subscript denotes a value in the current period. The absence of an m subscript in a symbol for which one is given denotes a value for the typical member patron.

b The base sum in the cooperative's tax bill.

C The set of outputs sold to and variable inputs purchased from the cooperative by the member and nonmember patrons.

C_{Dk} The amount of direct departmental costs allocated to the k-th department.

C_I The cooperative's total indirect departmental cost.

C_{Ik} The amount of indirect departmental costs allocated to the k-th department.

C_k The amount of indirect cost allocated to the k-th department (3.14).

D The determinant of the Jacobian matrix J.

D_{ij} The cofactor of the element in the i-th row and j-th column of the Jacobian matrix J.

D_k The subset X_k if k is a marketing department and Y_k if k is a supply department.

d The discount rate of the typical member patron.

d_c The discount rate used by the cooperative to discount allocated patronage refunds.

d_f The discount rate used by the cooperative to determine the present value of future profits.

DS The total dividend on member stock.

ds The dividends on stock held by the typical member patron.

f_i The average fixed cost allocated to the i-th product.

fc The fixed costs of the typical member patron or typical nonmember patron.

FCC The total fixed costs of the cooperative.

FCM The total fixed costs of the member patrons.

G The set of public goods provided by the cooperative.

I An instrument.

J The Jacobian matrix corresponding to the problem of the typical member patron (2.41).

L The Lagrangian function corresponding to the cooperative's problem of maximizing the total current profits of its member patrons (3.19).

M The number of member patrons in the cooperative association.

NS The total net savings of the cooperative (3.10).

NS_c The cooperative's net savings from member business.

NS_k The net savings of the k-th department (3.9).

NS_{kc} The net savings of the k-th department from member business.

NS_o The cooperative's net savings from nonmember business.

P_X A vector of the prices of the outputs in set X produced by the member and nonmember patrons.

P_Y A vector of the prices of the variable inputs in set Y used by the member and nonmember patrons.

p_i The price of the i-th product.

p_i^* The price or effective price of the i-th product.

PR The amount of patronage refunds allocated by the cooperative (3.4).

PVPR The present value of allocated patronage refunds.

pvpr The present value of the patronage refunds allocated to the typical member patron (2.2).

Q_G A vector of the quantities of each of the public goods in set G provided by the cooperative.

Q_V A vector of the quantities of each of the variable inputs in set V used by the cooperative and purchased from outside the cooperative association.

Q_{W_c} A vector of the quantities of each of the fixed inputs in set W_c used by the cooperative.

Q_X A vector of the quantities of each of the outputs in set X produced by the member and nonmember patrons and used by the cooperative.

Q_Y A vector of the quantities of each of the variable inputs in set Y purchased by the member and nonmember patrons.

Q_Z A vector of the quantities of each of the outputs in set Z produced by the cooperative and sold to buyers outside the cooperative association.

q_i The quantity of the i-th product.

q_{ic} The quantity of the i-th product purchased or sold by the member patrons.

q_{ij} The quantity of the i-th product used in the production of the j-th product.

q_{i0} The stock of the i-th fixed factor available.

q_{io} The quantity of the i-th product purchased or sold by the nonmember patrons.

q_{W_f} A vector of the quantities of each of the fixed inputs in set W_f used by the typical member patron.

q_X A vector of the quantities of each of the outputs in set X produced by the typical member patron.

q_Y A vector of the quantities of each of the variable inputs in set Y used by the typical member patron.

R The amount which must be taken out of the net savings from member business to meet the requirements of the cooperative (3.5).

R_c^p A vector of the past actual per-unit patronage refunds.

R_c^* A vector of the typical member patron's expected per-unit patronage refunds.

R_k The amount deducted from the net savings from member business of department k to meet the requirements of the cooperative.

r_i The per-unit patronage refund on the i-th product (3.11).

r_{im}^* The m-th member patron's expected per-unit patronage refund on the i-th product (2.3).

Req The amount which must be taken out of the net savings of the cooperative before patronage refunds can be allocated (3.6).

RS The retained savings which the cooperative adds to its surplus account.

S The subset of products in set C, the per-unit patronage refunds of which are affected by the instrument I.

s The proportion of allocated patronage refunds paid in cash.

s' $s + (1-s)/(1+d_c)^T$

T The length of the cooperative's time horizon.

TCC Total collective costs.

TCP Total collective profits.

TCR Total collective revenues.

TI_o The portion of the cooperative's taxable income for which the cooperative must pay the base sum b.

TMP Total member profits.

TPC Total private costs.

TPP Total private profits.

TPR Total private revenues.

TX The cooperative's total tax bill (3.3).

t The cooperative's marginal tax rate.

V The set of variable inputs used by the cooperative and purchased from outside the cooperative association.

v_i The average variable cost allocated to the i-th product.

W_c The set of fixed inputs available to the cooperative.

W_f The set of fixed factors available to the typical member patron.

X The set of outputs produced by the member and nonmember patrons.

X_c The subset of outputs in set X which are produced by the member and nonmember patrons and sold to the cooperative.

X_k The subset of products in set X handled by the k-th department.

X_o The subset of outputs in set X which are produced by the member and nonmember patrons and sold to buyers outside the cooperative association.

Y The set of variable inputs purchased by the member and nonmember patrons.

Y_c The subset of variable inputs in set Y purchased by the member and nonmember patrons from the cooperative.

Y_k The subset of products in set Y produced in the k-th department.

Y_o The subset of variable inputs in set Y purchased by the member and nonmember patrons from sellers outside the cooperative association.

Z The set of outputs produced by the cooperative and sold to buyers outside the cooperative association.

Z_k The subset products in set Z produced in the k-th department.

Γ The Lagrangian function corresponding to the cooperative's problem of maximizing the total discounted profits of its member patrons over its time horizon (5.2).

Δ The Lagrangian function corresponding to the problem of the typical nonmember patron (2.28).

δ_i The slack variable found in the i-th Kuhn-Tucker condition.

ε The elasticity of demand in the general market (4.20).

ε_o The elasticity of demand in the nonmember market (4.19).

Λ The Lagrangian function corresponding to the problem of the typical member patron (2.6).

λ_1 The Lagrange multiplier corresponding to the cooperative's production function.

λ_{2i} The Lagrange multiplier corresponding to the cooperative's i-th fixed-factor constraint.

λ_3 The Lagrange multiplier corresponding to the cooperative's nonmember-business constraint.

λ_4 The Lagrange multiplier corresponding to the cooperative's indirect-departmental-cost-allocation constraint.

η The elasticity of supply in the general market (4.15).

η_o The elasticity of supply in the nonmember market (4.14).

Π The total profits of the member patrons (3.1).

$\hat{\Pi}$ The total discounted profits of the member patrons (5.1).

π_m The profit of the m-th member patron (2.1).

ρ_k The ratio of the net savings of the k-th department to the total business of the k-th department (3.12).

σ The maximum proportion of nonmember business allowed by law.

σ_o The proportion of nonmember business (3.18).

τ The length of the cooperative's revolving fund.

Φ The implicit form of the production of the cooperative (3.2).

ϕ The implicit form of the production function of the typical member patron or typical nonmember patron (2.4).

ϕ_i The first-order partial derivative $\partial\phi/\partial q_i$.

ϕ_{ij} The second-order partial derivative $\partial^2\phi/\partial q_i \partial q_j$.

Ψ_1 The Lagrange multiplier corresponding to the production function of the typical member patron or typical nonmember patron.

Ψ_{2i} The Lagrange multiplier corresponding to the i-th fixed-factor constraint (2.5) of the typical member patron or typical nonmember patron.

APPENDIX B. PROOFS

Proof 1

This proof is intended to relate $-\Psi_1(\partial\phi/\partial q_k)$ for an output k to the marginal cost of producing the output.

The total differential of the production function ϕ is:

$$d\phi = \sum_{i\epsilon X} \frac{\partial\phi}{\partial q_i} dq_i + \sum_{i\epsilon Y} \frac{\partial\phi}{\partial q_i} dq_i + \sum_{i\epsilon W_c} \frac{\partial\phi}{\partial q_i} dq_i. \tag{A.1}$$

A change in total cost is:

$$dC_T = \sum_{i\epsilon Y} p_i^* \, dq_i + \sum_{i\epsilon W_f} \Psi_{2i} \, dq_i \tag{A.2}$$

where:

$p_i^* = p_i - s \cdot r_i^* - (1-s)\, r_i^*/(1+d)^T$ for all $i\epsilon Y_c$ in the case of a member patron,

$= p_i$ in the case of a nonmember patron and for all $i\epsilon Y_o$.

Setting:

$d\phi = 0$

$dq_i = 0$ for all $i\epsilon X$ except $i = k$,

$d\phi$ becomes:

$$d\phi = \frac{\partial\phi}{\partial q_k} dq_k + \sum_{i\epsilon Y} \frac{\partial\phi}{\partial q_i} dq_i + \sum_{i\epsilon W_f} \frac{\partial\phi}{\partial q_i} dq_i = 0. \tag{A.3}$$

Thus:

$$dq_k = -\frac{1}{\frac{\partial\phi}{\partial q_k}} \left[\sum_{i\epsilon Y} \frac{\partial\phi}{\partial q_i} dq_i + \sum_{i\epsilon W_f} \frac{\partial\phi}{\partial q_i} dq_i \right]. \tag{A.4}$$

Dividing A.2 by A.4, the marginal cost of producing output k is:

$$\frac{\partial C_T}{\partial q_k} = - \frac{\partial \phi}{\partial q_k} \left[\frac{\sum_{i \epsilon Y} p_i^* \, dq_i + \sum_{i \epsilon W_f} \Psi_{2i} \, dq_i}{\sum_{i \epsilon Y} \frac{\partial \phi}{\partial q_i} dq_i + \sum_{i \epsilon W_f} \frac{\partial \phi}{\partial q_i} dq_i} \right] . \tag{A.5}$$

From the Kuhn-Tucker conditions:

$$p_i^* = \Psi_1 \frac{\partial \phi}{\partial q_i} + \delta_i \text{ for all } i \epsilon Y \tag{A.6}$$

and:

$$\Psi_{2i} = \Psi_1 \frac{\partial \phi}{\partial q_i} + \delta_i \text{ for all } i \epsilon W_f \tag{A.7}$$

where:

$$\delta_i > 0 \text{ if } q_i = 0$$

$$\delta_i = 0 \text{ if } q_i > 0.$$

Substituting A.6 and A.7 into A.5, it becomes:

$$\frac{\partial C_T}{\partial q_k} = - \frac{\partial \phi}{\partial q_k} \left[\frac{\sum_{i \epsilon Y, W_f} (\Psi_1 \frac{\partial \phi}{\partial q_i} + \delta_i) \, dq_i}{\sum_{i \epsilon Y, W_f} \frac{\partial \phi}{\partial q_i} dq_i} \right] . \tag{A.8}$$

If it is assumed that no factor will come into use that is not already in use, either δ_i or dq_i will equal zero for all $i \epsilon Y, W_f$ and the term within the brackets reduces to Ψ_1. Thus, the marginal cost of producing output k is:

$$\frac{\partial C_T}{\partial q_k} = - \Psi_1 \frac{\partial \phi}{\partial q_k} . \qquad \text{Q.E.D.} \tag{A.9}$$

Proof 2

This proof is intended to relate $\Psi_1(\partial\phi/\partial q_i)$ for an input k to the marginal value product of the input.

A change in total revenue is:

$$dR_T = \sum_{i\epsilon X} p_i^* \, dq_i \tag{A.10}$$

where:

$p_i^* = p_i + s \cdot r_i^* + (1-s)\, r_i^*/(1+d)^T$ for all $i\epsilon X_c$ in the case of a member patron,

$= p_i$ in the case of a nonmember patron and for all $i\epsilon X_o$.

Setting:

$d\phi = 0$

$dq_i = 0$ for all $i\epsilon Y$ and $i\epsilon W_f$ except $i = k$,

$d\phi$ becomes:

$$d\phi = \frac{\partial\phi}{\phi q_k} dq_k + \sum_{i\epsilon X} \frac{\partial\phi}{\partial q_i} dq_i = 0. \tag{A.11}$$

Thus:

$$dq_k = - \frac{1}{\dfrac{\partial\phi}{\partial q_k}} \left[\sum_{i\epsilon X} \frac{\partial\phi}{\partial q_i} dq_i \right]. \tag{A.12}$$

Dividing A.9 by A.11, the marginal revenue product of input k is:

$$\frac{\partial R_T}{\partial q_k} = - \frac{\partial\phi}{\partial q_k} \left[\frac{\sum_{i\epsilon X} p_i^* \, dq_i}{\sum_{i\epsilon X} \dfrac{\partial\phi}{\partial q_i} dq_i} \right]. \tag{A.13}$$

From the Kuhn-Tucker conditions:

$$p_i^* = - \Psi_1 \frac{\partial\phi}{\partial q_i} - \delta_i \quad \text{for all } i\epsilon X \tag{A.14}$$

where:

$$\delta_i > 0 \text{ if } q_i = 0$$

$$\delta_i = 0 \text{ if } q_i > 0.$$

Substituting A.14 into A.13, it becomes:

$$\frac{\partial R_T}{\partial q_k} = \frac{\partial \phi}{\partial q_k} \left[\frac{\sum_{i \epsilon X} \left(\Psi_1 \frac{\partial \phi}{\partial q_i} + \delta_i \right) dq_i}{\sum_{i \epsilon X} \frac{\partial \phi}{\partial q_i} dq_i} \right] . \qquad (A.15)$$

If it is assumed that no product will come into production that is not already in production, either δ_i or dq_i will be equal to zero for all $i \epsilon X$ and the term within the brackets reduces to Ψ_1. Thus, the marginal revenue product or, in this case, the marginal value product of input k is:

$$\frac{\partial R_T}{\partial q_k} = \Psi_1 \frac{\partial \phi}{\partial q_k} . \qquad \text{Q.E.D.} \qquad (A.16)$$

Proof 3

This proof is intended to relate $-\lambda_1 (\partial \Phi / \lambda q_k)$, where $k \epsilon Y_c$, to the marginal variation in the profits of the member patrons arising from a change in the quantity of output k produced by the cooperative and sold to patrons.

The total differential of the cooperative's production function Φ is:

$$d\Phi = \sum_{i \epsilon C,G,Z} \frac{\partial \Phi}{\partial q_i} dq_i + \sum_{i \epsilon V,W_c} \sum_{j \epsilon G,Y_c,Z} \frac{\partial \Phi}{\partial q_{ij}} dq_{ij} . \qquad (A.17)$$

A change in the total profits of the member patrons is:

$$d\Pi = \sum_{i\epsilon Z} \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i\epsilon C,G} \left(\frac{\partial \Pi}{\partial q_i} - \lambda_3 \frac{\partial \sigma_o}{\partial q_i}\right) dq_i$$

$$+ \sum_{i\epsilon V} \sum_{j\epsilon G,Y_c,Z} \frac{\partial \Pi}{\partial q_{ij}} dq_{ij} - \sum_{i\epsilon W_c} \sum_{j\epsilon G,Y,Z} \lambda_{2i}\, dq_{ij}. \qquad (A.18)$$

Setting:

$$d\bar{\Phi} = 0$$

$$dq_{ij} = 0 \quad \text{for all } j\epsilon G,Z$$

and limiting dq_i for $i\epsilon C$, $i\neq k$, to the change in the quantity used directly or indirectly in producing k, the marginal variation in the profits of the member patrons arising from a change in the quantity of output k produced by the cooperative and sold to patrons is:

$$d\Pi = \sum_{\substack{i\epsilon C \\ i\neq k}} \left(\lambda_3 \frac{\partial \sigma_o}{\partial q_i} - \frac{\partial \Pi}{\partial q_i}\right) dq_i - \sum_{i\epsilon V} \sum_{j\epsilon Y_c} \frac{\partial \Pi}{\partial q_{ij}} dq_{ij}$$

$$+ \sum_{i\epsilon W_c} \sum_{j\epsilon Y_c} \lambda_{2i}\, dq_{ij}. \qquad (A.19)$$

$d\bar{\Phi}$ becomes:

$$d\bar{\Phi} = \frac{\partial \bar{\Phi}}{\partial q_k} dq_k + \sum_{\substack{i\epsilon C \\ i\neq k}} \frac{\partial \bar{\Phi}}{\partial q_i} dq_i + \sum_{i\epsilon V,W_c} \sum_{j\epsilon Y_c} \frac{\partial \bar{\Phi}}{\partial q_{ij}} dq_{ij} = 0. \qquad (A.20)$$

Thus:

$$dq_k = -\frac{1}{\frac{\partial \bar{\Phi}}{\partial q_k}} \left[\sum_{\substack{i\epsilon C \\ i\neq k}} \frac{\partial \bar{\Phi}}{\partial q_i} dq_i + \sum_{i\epsilon V,W_c} \sum_{j\epsilon Y_c} \frac{\partial \bar{\Phi}}{\partial q_{ij}} dq_{ij} \right]. \qquad (A.21)$$

Dividing A.19 by A.21, the marginal variation in the profits of the member patrons arising from a change in the quantity of output k produced by the cooperative and sold to patrons is:

$$\frac{\partial \Pi}{\partial q_k} = -\frac{\partial \Phi}{\partial q_k}\left[\frac{\sum_{\substack{i\in C\\ i\neq k}}\left(\lambda_3\frac{\partial \sigma_o}{\partial q_i}-\frac{\partial \Pi}{\partial q_i}\right)dq_i \;-\; \sum_{i\in V}\sum_{j\in Y_c}\frac{\partial \Pi}{\partial q_{ij}}dq_{ij}}{\sum_{\substack{i\in C\\ i\neq k}}\frac{\partial \Phi}{\partial q_i}dq_i \;+\; \sum_{i\in V}\sum_{j\in Y_c}\frac{\partial \Phi}{\partial q_{ij}}dq_{ij}}\right.$$

$$\left.\frac{+\sum_{i\in W_c}\sum_{j\in Y_c}\lambda_{2i}\,dq_{ij}}{+\sum_{i\in W_c}\sum_{j\in Y_c}\frac{\partial \Phi}{\partial q_{ij}}dq_{ij}}\right]. \qquad \text{(A.22)}$$

From Kuhn-Tucker conditions 3.24 and 3.25:

$$\frac{\partial \Pi}{\partial q_{ij}} = -\lambda_1\frac{\partial \Phi}{\partial q_{ij}} - \delta_{ij} \quad \text{for all } i\in V;\ j\in Y_c \qquad \text{(A.23)}$$

and:

$$\lambda_{2i} = \lambda_1\frac{\partial \Phi}{\partial q_{ij}} + \delta_{ij} \quad \text{for all } i\in W_c;\ j\in Y_c \qquad \text{(A.24)}$$

where:

$$\delta_{ij} > 0 \text{ if } q_{ij} = 0$$

$$\delta_{ij} = 0 \text{ if } q_{ij} > 0.$$

Also, by using q_j as an instrument instead of p_j, Kuhn-Tucker conditions 3.20 and 3.21 can be replaced by:

$$\lambda_3\frac{\partial \sigma_o}{\partial q_j} - \frac{\partial \Pi}{\partial q_j} = \lambda_1\frac{\partial \Phi}{\partial q_j} + \delta_j \quad \text{for all } i\in C \qquad \text{(A.25)}$$

where:

$$\delta_j > 0 \text{ if } q_j = 0$$

$$\delta_j = 0 \text{ if } q_j > 0.$$

Substituting A.23 through A.25 into A.22, it becomes:

$$\frac{\partial \Pi}{\partial q_k} = - \frac{\partial \Phi}{\partial q_k} \left[\frac{\sum_{\substack{i \in C \\ i \neq k}} (\lambda_1 \frac{\partial \Phi}{\partial q_i} + \delta_i)\, dq_i}{\sum_{\substack{i \in C \\ i \neq k}} \frac{\partial \Phi}{\partial q_i}\, dq_i} + \frac{\sum_{i \in V, W_c} \sum_{j \in Y_c} (\lambda_1 \frac{\partial \Phi}{\partial q_{ij}} + \delta_{ij})\, dq_{ij}}{\sum_{i \in V, W_c} \sum_{j \in Y_c} \frac{\partial \Phi}{\partial q_{ij}}\, dq_{ij}} \right] . \qquad (A.26)$$

If it is assumed that no factor will come into use that is not already in use, either δ_i or dq_i will equal zero for all $i \in C$ and either δ_{ij} or dq_{ij} will equal zero for all $i \in V, W_c$; $j \in Y_c$ and the term within the brackets reduces to λ_1. Thus, the marginal variation in the profits of the member patrons arising from a change in the quantity of output k produced by the cooperative and sold to patrons is:

$$\frac{\partial \Pi}{\partial q_k} = - \lambda_1 \frac{\partial \Phi}{\partial q_k} \quad . \qquad \text{Q.E.D.} \qquad (A.27)$$

Proof 4

This proof is intended to relate $\lambda_1 (\partial \tilde{\Phi} / \partial q_k)$, where $k \in X$, to the marginal variation in the profits of the member patrons arising from a change in the quantity of input k obtained from patrons and used in production by the cooperative.

Setting:

$d\Phi = 0$

$dq_i = 0$ for all $i \in X$ except $i=k$

$dq_{ij} = 0$ for all $i \in V, W_c$

and limiting dq_i for $i \epsilon Y_c$ to the change in the quantity not used directly or indirectly in production, the marginal variation in the profits of the member patrons arising from a change in the quantity of input k obtained from patrons and used in production by the cooperative is:

$$d\Pi = \sum_{i \epsilon Z} \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i \epsilon G, Y_c} \left(\frac{\partial \Pi}{\partial q_i} - \lambda_3 \frac{\partial \sigma_o}{\partial q_i} \right) dq_i . \tag{A.28}$$

$d\Phi$ becomes:

$$d\Phi = \frac{\partial \Phi}{\partial q_k} dq_k + \sum_{i \epsilon Z} \frac{\partial \Phi}{\partial q_i} dq_i + \sum_{i \epsilon G, Y_c} \frac{\partial \Phi}{\partial q_i} dq_i = 0. \tag{A.29}$$

Thus:

$$dq_k = - \frac{1}{\frac{\partial \Phi}{\partial q_k}} \left[\sum_{i \epsilon Z} \frac{\partial \Phi}{\partial q_i} dq_i + \sum_{i \epsilon G, Y_c} \frac{\partial \Phi}{\partial q_i} dq_i \right] . \tag{A.30}$$

Dividing A.28 by A.30, the marginal variation in the profits of the member patrons arising from a change in the quantity of input k obtained from patrons and used in production by the cooperative is:

$$\frac{\partial \Pi}{\partial q_k} = - \frac{\partial \Phi}{\partial q_k} \left[\frac{\sum_{i \epsilon Z} \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i \epsilon G, Y_c} \left(\frac{\partial \Pi}{\partial q_i} - \lambda_3 \frac{\partial \sigma_o}{\partial q_i} \right) dq_i}{\sum_{i \epsilon Z} \frac{\partial \Phi}{\partial q_i} dq_i + \sum_{i \epsilon G, Y_c} \frac{\partial \Phi}{\partial q_i} dq_i} \right] . \tag{A.31}$$

From Kuhn-Tucker condition 3.22 and 3.23:

$$\lambda_3 \frac{\partial \sigma_o}{\partial q_i} - \frac{\partial \Pi}{\partial q_i} = \lambda_1 \frac{\partial \Phi}{\partial q_i} + \delta_i \quad \text{for all } i \epsilon G \tag{A.32}$$

and:

$$\frac{\partial \Pi}{\partial q_i} = - \lambda_1 \frac{\partial \Phi}{\partial q_i} + \delta_i \quad \text{for all } i \epsilon Z \tag{A.33}$$

where:

$$\delta_i > 0 \text{ if } q_i = 0$$

$$\delta_i = 0 \text{ if } q_i > 0.$$

Substituting A.25, A.32, and A.33 into A.31, it becomes:

$$\frac{\partial \Pi}{\partial q_k} = \frac{\partial \Phi}{\partial q_k} \left[\frac{\sum_{i \in G, Y_c, Z} (\lambda_1 \frac{\partial \Phi}{\partial q_i} + \delta_i)\, dq_i}{\sum_{i \in G, Y_c, Z} \frac{\partial \Phi}{\partial q_i}\, dq_i} \right]. \tag{A.34}$$

If it is assumed that no product will come into production that is not already in production, either δ_i or q_i will equal zero for all $i \in G, Y_c, Z$ and the term within the brackets reduces to λ_1. Thus, the marginal variation in the profits of the member patrons arising from a change in the quantity of input k obtained from patrons and used in production by the cooperative is:

$$\frac{\partial \Pi}{\partial q_k} = \lambda_1 \frac{\partial \Phi}{\partial q_k}. \qquad \text{Q.E.D.} \tag{A.35}$$

Proof 5

Define net profits as:

$$\Pi_n = \Pi - T_s - T_f \tag{A.36}$$

where T_s represents state income tax and T_f represents federal income tax. These can be written:

$$T_s = t_s(\Pi - T_f) \tag{A.37}$$

and:

$$T_f = t_f(\Pi - T_s) \tag{A.38}$$

where t_s and t_f represent the state and federal marginal tax rates respectively.

If the state marginal rate is a constant α times the federal rate, A.37 can be rewritten:

$$T_s = \alpha t_f(\Pi - T_f). \tag{A.39}$$

Substituting A.38 into A.39, it becomes:

$$T_s = \alpha t_f \Pi - \alpha t_f^2 \Pi + \alpha t_f^2 T_s. \tag{A.40}$$

Solving A.40 for T_s, it becomes:

$$T_s = \frac{\alpha(t_f - t_f^2)\Pi}{(1-\alpha t_f^2)} . \tag{A.41}$$

Substituting A.41 into A.38, A.38 can be expressed:

$$T_f = t_f[\Pi - \frac{\alpha(t_f - t_f^2)\Pi}{(1-\alpha t_f^2)}] . \tag{A.42}$$

Substituting both A.41 and A.42 into A.36, net profits is equivalent to:

$$\Pi_n = [1 - t_f + \frac{\alpha(t_f-1)(t_f-t_f^2)}{(1-\alpha t_f^2)}]\Pi. \tag{A.43}$$

This can be rewritten:

$$\Pi_n = (1 - t)\Pi \tag{A.44}$$

where:

$$t = t_f - \frac{\alpha(t_f-1)(t_f-t_f^2)}{(1-\alpha t_f^2)} . \tag{A.45}$$